UDC 528.236.4

CONSTRUCTION OF THE PLANE OF THE NORMAL SECTION FOR TRIGONOMETRIC LEVELING BY GNSS SPATIAL MEASUREMENTS

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Keywords: vector product, GNSS measurement, ellipsoid of rotation, canonical equations, disjoint directs, mixed product, normal, normal plane, parametric equations, plane of normal section, WGS-84.

Formulation of the problem. It is known that the basic formulas of geometric and trigonometric leveling are obtained in the plane of a normal section, which passes through points, the excess between which is determined [1]. But spatial measurements of GNSS at appropriate observing stations are carried out in different planes, which leads to inevitable errors in their mathematical elaboration. Therefore, the relevance of this topic is the need to construct such a normal plane, on which it is possible to carry out all calculations of the results of observations obtained from different points of GNSS.

Setting objectives. Construct the plane of a normal section and compute on it the coordinates of the point of intersection of two normal and the angle between them for trigonometric alignment for spatial measurements of GNSS.

Presenting main material. Let the desired plane passes through a point $B(x_2, y_2, z_2)$ and a straight line s_1 , given by canonical equations [4]

$$\frac{x-x_1}{m_1}=\frac{y-y_1}{n_1}=\frac{z-z_1}{p_1} \quad . \tag{1}$$

Then vectors $\vec{AM}$, $\vec{AB}$ and $\vec{s_1}=\{m_1, n_1, p_1\}$ - are complicated. Here is $A(x_1, y_1, z_1)$ - the point, belonging to the line s_1; $M(x, y, z)$ - arbitrary point of the plane.

That is, the mixed product $(\vec{AM}, \vec{AB}, \vec{s_1})=0$ or in the coordinate system

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ m_1 & n_1 & p_1 \end{vmatrix}=0 \quad . \tag{2}$$

If you accept the following notation:

$$\Delta x_1=x-x_1; \quad \Delta y_1=y-y_1; \quad \Delta z_1=z-z_1;$$
$$\Delta x_{21}=x_2-x_1; \quad \Delta y_{21}=y_2-y_1; \quad \Delta z_{21}=z_2-z_1,$$

then the mixed product (2) can be shown as follows:

$$\begin{vmatrix} \Delta x_1 & \Delta y_1 & \Delta z_1 \\ \Delta x_{21} & \Delta y_{21} & \Delta z_{21} \\ m_1 & n_1 & p_1 \end{vmatrix}=0 \quad . \tag{3}$$

Then the equation of the plane, which we denote by π_1 , will be written in the form:

$$A_1x+B_1y+C_1z+D_1=0 \quad , \tag{4}$$

where

$$A_1=\Delta y_{21}\cdot p_1-\Delta z_{21}\cdot n_1;$$
$$B_1=\Delta z_{21}\cdot m_1-\Delta x_{21}\cdot p_1;$$
$$C_1=\Delta x_{21}\cdot n_1-\Delta y_{21}\cdot m_1;$$
$$D_1=-(A_1x_1+B_1y_1+C_1z_1).$$

It should be noted that the plane π_1 will be called the *plane of the normal section*, since it passes through the straight s_1 , which is normal to the ellipsoid of rotation, and the point $B(x_2, y_2, z_2)$ on the ellipsoid [1].

It is shown in [2] that two straight lines s_1 and s_2 (which are normal to the ellipsoid of rotation, carried out from two points of observation) do not intersect, except if these points are located on one meridian.

To find a common point of intersection of two normals, which in reality are disjoint directs [2], it is necessary to make a projection of a straight line s_2 (a straight line that does not belong to the plane π_1) onto a plane of normal section [3].

To do this, we first construct a plane that passes through a given straight line s_2 perpendicular to the plane of the normal section π_1.

Let the plane π_2 pass through the straight line s_2 , given by canonical equations [4]

$$\frac{x-x_2}{m_2}=\frac{y-y_2}{n_2}=\frac{z-z_2}{p_2} \quad , \tag{5}$$

perpendicular to the plane π_1 , shown by equation (4).

Then the vectors $\vec{BM}$, $\vec{s_2}=\{m_2, n_2, p_2\}$ and $\vec{n}$ - are complicated, that is, the mixed product $(\vec{BM}, \vec{s_2}, \vec{n})=0$ or

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ m_2 & n_2 & p_2 \\ A_1 & B_1 & C_1 \end{vmatrix}=0 \quad . \tag{6}$$

If the following notation:

$$\Delta x_2=x-x_2; \quad \Delta y_2=y-y_2; \quad \Delta z_2=z-z_2,$$

then the mixed product (6) can be represented as follows:

$$\begin{vmatrix} \Delta x_2 & \Delta y_2 & \Delta z_2 \\ m_2 & n_2 & p_2 \\ A_1 & B_1 & C_1 \end{vmatrix}=0 \quad . \tag{7}$$

Then the equation of plane π_2, perpendicular to the plane of the normal section π_1, is written as:

$$A_2x+B_2y+C_2z+D_2=0 \quad , \tag{8}$$

where

$$A_2 = C_1 \cdot n_2 - B_1 \cdot p_2;$$
$$B_2 = A_1 \cdot p_2 - C_1 \cdot m_2;$$
$$C_2 = B_1 \cdot m_2 - A_1 \cdot n_2;$$
$$D_2 = -(A_2 x_2 + B_2 y_2 + C_2 z_2).$$

Now we find the equation of direct s_3 as a line of intersection of two perpendicular planes π_1 and π_2 .

It is known that a straight line can be regarded as a line of intersection of two nonparallel planes, that is, a straight line in space can be specified by a system of two linear equations [4]:

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 , \\ A_2 x + B_2 y + C_2 z + D_2 = 0 . \end{cases} \quad (9)$$

Since the straight line s_3 is perpendicular to the normal vectors $\vec{n_1} = \{A_1, B_1, C_1\}$ and $\vec{n_2} = \{A_2, B_2, C_2\}$, then the direct vector $\vec{s_3}$ of the line can take the vector product $[\vec{n_1}, \vec{n_2}]$, that is

$$\vec{s_3} = [\vec{n_1}, \vec{n_2}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} . \quad (10)$$

Where

$$m_3 = B_1 \cdot C_2 - C_1 \cdot B_2;$$
$$n_3 = C_1 \cdot A_2 - A_1 \cdot C_2;$$
$$p_3 = A_1 \cdot B_2 - B_1 \cdot A_2.$$

Since the point $B(x_2, y_2, z_2)$ belongs to the straight line s_3 , then the canonical equations of the straight line s_3 have the form

$$\frac{x - x_2}{m_3} = \frac{y - y_2}{n_3} = \frac{z - z_2}{p_3} . \quad (11)$$

The resulting line s_3 is a projection of the normal s_2 onto the plane π_1 .

To find the intersection point of two straight s_1 and s_3 , that belong to the plane of the normal section, we need to solve the following system of equations:

$$\begin{cases} \dfrac{x - x_1}{m_1} = \dfrac{y - y_1}{n_1} = \dfrac{z - z_1}{p_1}; \\ \dfrac{x - x_2}{m_3} = \dfrac{y - y_2}{n_3} = \dfrac{z - z_2}{p_3}. \end{cases} \quad (12)$$

Having written the straight line s_1 in the form of parametric equations and substituting the canonical equations of direct s_3 , we get

$$\begin{cases} (n_3 m_1 - n_1 m_3) \cdot t - n_3 \Delta x_{21} + m_3 \Delta y_{21} = 0; \\ (p_3 m_1 - p_1 m_3) \cdot t - p_3 \Delta x_{21} + m_3 \Delta z_{21} = 0. \end{cases} \quad (13)$$

Where

$$t = \frac{n_3 \Delta x_{21} - m_3 \Delta y_{21}}{n_3 m_1 - n_1 m_3} \text{ або } t = \frac{p_3 \Delta x_{21} - m_3 \Delta z_{21}}{p_3 m_1 - p_1 m_3} \quad (14)$$

Thus, the rectangular coordinates of the point $P(x_P, y_P, z_P)$ of intersection of two straight lines belonging to the plane of the normal section can be calculated by the formula:

$$\begin{cases} x_P = x_1 + m_1 \cdot t; \\ y_P = y_1 + n_1 \cdot t; \\ z_P = z_1 + p_1 \cdot t. \end{cases} \quad (15)$$

We find the angle ψ between these lines by the formula of the scalar product of two vectors $\vec{s_1}$ and $\vec{s_3}$ [4]

$$\cos\psi = \frac{(\vec{s_1}, \vec{s_3})}{|\vec{s_1}| \cdot |\vec{s_3}|} , \quad (16)$$

where

$$(\vec{s_1}, \vec{s_3}) = m_1 m_3 + n_1 n_3 + p_1 p_3 \quad (17)$$

– scalar product of vectors $\vec{s_1}$ and $\vec{s_3}$;

$$|\vec{s_1}| = \sqrt{m_1^2 + n_1^2 + p_1^2}; \quad (18)$$

$$|\vec{s_3}| = \sqrt{m_3^2 + n_3^2 + p_3^2} \quad (19)$$

– lengths of vectors $\vec{s_1}$ and $\vec{s_3}$ respectively.

Substituting equation (17) – (19) into expression (16), we obtain the formula for calculating the angle between the straight lines that belong to the plane of the normal section, that is

$$\cos\psi = \frac{m_1 m_3 + n_1 n_3 + p_1 p_3}{\sqrt{m_1^2 + n_1^2 + p_1^2} \cdot \sqrt{m_3^2 + n_3^2 + p_3^2}} \quad (20)$$

The magnitude of the angle ψ depends on the distance between the points of observation and carries information about the curvature of the Earth and replaces the value of the mean radius of the curvature of the Earth in the trigonometric leveling.

The results of computing the rectangular coordinates of the point of intersection $P(x_P, y_P, z_P)$ of two straight lines belonging to the constructed plane of the normal section, as well as the value of the angle ψ between them, are given in Table 1.

It should be noted that the calculations given in this table used GNSS spatial measurements at two stations for the period 2012-2017, and the plane of the normal section was constructed at the normal point at the $A(x_1, y_1, z_1)$ point to the WGS-84 rotation ellipsoid and the $B(x_2, y_2, z_2)$ point of the same ellipsoid.

Table 1

Spatial measurements of GNSS on a plane of normal section

Items GNSS measurement	Point coordinates on the normal plane (m)			Angle (")
	X_P	Y_P	Z_P	ψ
A – *B* (2012)	1457.567	757.803	-30186.249	32.3722
A – *B* (2013)	1457.565	757.802	-30186.251	32.3721
A – *B* (2014)	1457.563	757.801	-30186.254	32.3721
A – *B* (2015)	1457.578	757.809	-30186.234	32.3721
A – *B* (2016)	1457.537	757.788	-30186.287	32.3721
A – *B* (2017/1)	1457.583	757.812	-30186.228	32.3720
A – *B* (2017/2)	1457.573	757.806	-30186.241	32.3719

Conclusions. By normals to the WGS-84 rotation ellipsoid, in the GNSS observations, the plane of the normal section was constructed and working formulas were obtained for finding the rectangular coordinates of the point of intersection of two straight lines belonging to the plane of the normal section, as well as formulas for determining the angle between them.

According to GNSS spatial measurements, two observation stations for the period 2012-2017 obtained the results of computing rectangular coordinates of the intersection point of the $P(x_P, y_P, z_P)$ of two straight lines belonging to the constructed plane of the normal section and the angle ψ between these lines.

Literature

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Construction of the plane of the normal section for trigonometric leveling by GNSS spatial measurements

The plane of the normal section is constructed and the results of calculations of the rectangular coordinates of the point of intersection of the two straight lines belonging to this plane are obtained, as well as the angle value between the given lines.