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RELATIONSHIP BETWEEN GLOBAL AND REGIONAL GRAVITATIONAL FIELD

B.B. Dzhuman

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Problem statement

Beginning with the fundamental studies by Legendre and Gauss in the theory of Newtonian potentials, the classical representation of the gravitational field became its notation in the form of infinite series of Legendre spherical functions. However, in the case of a regional gravitational field, in a limited region, the Legendre spherical functions lose their orthogonality and the solution becomes unstable. For the modeling of a regional magnetic field in 1985, the method of spherical cap harmonic analysis (SCHA) [7-9] was proposed, which is based on the using of spherical Legendre functions of integer degree and real order. Such functions are the solution of the Laplace equation in the sphere segment and form two orthogonal systems of functions in a given region. In turn, in the articles [2,3,5,6,12] this method has been adapted to the problems of geodesy, in particular, for modeling of regional gravitational potential.

In 1992, the method of ASHA [1,4,13] was developed on the basis of the SCHA method, which involves replacing the coordinate system and the transition from the "cap" of the sphere to the hemisphere, which makes it possible to use the Legendre spherical functions of integer degree and real order to model the regional field.

Task statement

Because of great interest of the transition from regional to global model coefficients, in [14] was found this transition from the SCHA coefficients to global coefficients, and in work [15] - from the ASHA coefficients to global coefficients. However, SCHA-functions and global spherical functions have a different area of definition, and the transition of coefficients have no analytical form, but are more likely empirical. In [15], a transition analogous to the perforation in [14] was found, which is not appropriate, since the ASHA-functions and global spherical functions have the same area of definition. Accordingly, in this paper we will consider the analytical transition from ASHA-coefficients to global model coefficients.

Presentation of the main research material

Almost any function defined on the sphere, including gravitational potential *V*, can be expanded in a series of spherical Legendre functions

$$V(\theta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(C_{nm}\cos m\lambda+S_{nm}\sin m\lambda)P_{nm}(\cos\theta), \qquad (1)$$

where $P_{nm}(\cos\theta)$ is spherical Legendre functions of the first kind, C_{nm} and S_{nm} is unknown global coefficients, θ,λ is spherical coordinates.

In turn, when modeling a function given on a spherical segment $\theta\le\theta_0$, using Legendre's global spherical functions leads to an unsustainable solution. Using the Sturm-Liouville theorem it is easy to establish that on a spherical segment spherical Legendre functions of integer degree and real order are harmonic $P_{n_k m}(\cos\theta)$, where k is ordering index. Such functions form two orthogonal systems of functions, but in general they are not orthogonal. Extending a function on a sphere segment in a row for such functions, we obtain

$$\tilde{V}(\theta,\lambda)=\sum_{k=0}^{\infty}\sum_{m=0}^{k}(\tilde{C}_{km}\cos m\lambda+\tilde{S}_{km}\sin m\lambda)P_{n_k m}(\cos\theta), \qquad (2)$$

where $\tilde{C}_{nm}$ and $\tilde{S}_{nm}$ is unknown regional SCHA-coefficients.

In paper [4] the ASHA method for modeling a regional field is proposed. It involves transforming the initial data from the sphere segment into hamisphere using a simple relationship

$$\left.\begin{aligned} r' &= r,\\ \lambda' &= \lambda,\\ \theta' &= s\cdot\theta, \end{aligned}\right\} \qquad (3)$$

where $s=\dfrac{\pi}{2\theta_0}$, r',λ',θ' is new spherical coordinate system. This approach allows us to use the Legendre spherical functions of integer degree and order, which, unlike SCHA-functions, can be

decomposed into a finite hypergeometric series, and it is much easier to calculate the "full" norm of functions unlike the norm of SCHA-functions [10,11]. However, it should be noted that in the half-sphere the Legendre function form two systems of orthogonal functions and in general they are not orthogonal. So, in this case, the function $\overline{V}$, defined on the cap of the sphere, can be expanded in a row

$$\overline{V}(\theta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(C_{nm}^{*}\cos m\lambda+S_{nm}^{*}\sin m\lambda)P_{nm}(\cos s\theta), \quad (4)$$

where C_{nm}^{*} and S_{nm}^{*} are unknown regional ASHA-coefficients.

In [14] the transition from the SCHA-coefficients to global coefficients is considered in detail. However, the SCHA-functions and global spherical Legendre functions have different definition areas. In turn, it is obvious that the ASHA-functions have the same definition area as the global spherical functions. Consider the transition from ASHA-coefficients to global coefficients.

Let us imagine that some function V is given at every point of the sphere segment $\theta<\theta_0$, $r=1$. In this case it can be completely precisely modeled, expanded in series by the global Legendre spherical functions

$$V=\sum_{n=0}^{\infty}\sum_{m=0}^{n}[C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)]P_{nm}(\cos\theta). \quad (5)$$

On the other hand it can also be completely precisely expanded into a row using ASHA-functions

$$\overline{V}(\theta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(C_{nm}^{*}\cos m\lambda+S_{nm}^{*}\sin m\lambda)P_{nm}(\cos s\theta). \quad (6)$$

Since the models (5) and (6) represent the function absolutely precisely, there is an identity

$$V\equiv\overline{V}. \quad (7)$$

Equating the right-hand sides of equations (5) and (6) and multiplying them from both sides on $\overline{P}_{ij}(\cos\theta)\cos(j\lambda)$, we obtain

$$P_{ij}(\cos\theta)\cos(j\lambda)\sum_{n=0}^{\infty}\sum_{m=0}^{n}[C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)]P_{nm}(\cos\theta)= \quad (8)$$
$$P_{ij}(\cos\theta)\cos(j\lambda)\sum_{n=0}^{\infty}\sum_{m=0}^{n}[C_{nm}^{*}\cos(m\lambda)+S_{nm}^{*}\sin(m\lambda)]P_{nm}(\cos s\theta).$$

Integrating (8) on both sides of the sphere and using the orthogonal properties of the Legendre spherical functions, after minor mathematical transformations we obtain

$$\left.\begin{aligned} C_{ij}&=\frac{1+\delta_{j0}}{4}\sum_{n=j}^{\infty}C_{nj}^{*}\int_{0}^{\pi}P_{nj}(\cos s\theta)P_{ij}(\cos\theta)\sin\theta d\theta \quad for\ m=0;\\ \begin{matrix}C_{ij}\\S_{ij}\end{matrix}&=\frac{1}{4}\sum_{n=j}^{\infty}\begin{matrix}C_{nj}^{*}\\S_{nj}^{*}\end{matrix}\int_{0}^{\pi}P_{nj}(\cos s\theta)P_{ij}(\cos\theta)\sin\theta d\theta \quad for\ m\neq 0.\end{aligned}\right\} \quad (9)$$

Let us enter the notation

$$\alpha_{ijk}(s)=\int_{0}^{\pi}P_{kj}(\cos s\theta)P_{ij}(\cos\theta)\sin\theta d\theta. \quad (10)$$

Then formula (9) can be rewritten in the following form

$$\left.\begin{aligned} C_{ij}&=\frac{1+\delta_{j0}}{4}\sum_{n=j}^{\infty}C_{nj}^{*}\alpha_{ijn} \quad для\ \ m=0;\\ \begin{matrix}C_{ij}\\S_{ij}\end{matrix}&=\frac{1}{4}\sum_{n=j}^{\infty}\frac{C_{nj}^{*}}{S_{nj}^{*}}\alpha_{ijn} \quad для\ \ m\neq 0.\end{aligned}\right\} \quad (11)$$

In the case when the function is specified on the segment $\theta_0=90°$, ie parameter $s=1$, equations (10) and (11) show that equality is present

$$\begin{matrix}C_{ij}\\S_{ij}\end{matrix}=\begin{matrix}C_{ij}^{*}\\S_{ij}^{*}\end{matrix}. \quad (12)$$

The type of integral (10) is the Fredholm kernel of the first kind. Since the general parameter *s* is a real number, the solution of such an integral is rather cumbersome. For example, even for low orders we get

$$\alpha_{100}(s)=-\frac{\sqrt{3}}{2}\left[\frac{1}{1-s}\cos((1-s)\pi)+\frac{1}{1+s}\cos((1+s)\pi)-\frac{1}{1-s}-\frac{1}{1+s}\right]. \quad 13)$$

To find such integrals, methods of approximate integration can be used. For example, in Table 1 the value $\alpha_{ijk}(s)$ is given using such methods for the sphere segment $\theta_0=25°$ (*s*=3.6).

Table 1

Approximate value of integrals $\alpha_{ijk}(s)$ for $\theta_0=25°$

	$P_{00}(\cos s\theta)$	$P_{10}(\cos s\theta)$	$P_{20}(\cos s\theta)$	$P_{30}(\cos s\theta)$	$P_{40}(\cos s\theta)$
$P_{00}(\cos\theta)$	2,000	-0,190	1,112	-0,111	0,830
$P_{10}(\cos\theta)$	0,000	-0,231	-0,110	-0,179	-0,071
$P_{20}(\cos\theta)$	0,000	-1,387	-0,017	-0,801	-0,034
$P_{30}(\cos\theta)$	0,000	1,390	-0,238	0,716	-0,149
$P_{40}(\cos\theta)$	0,000	1,385	-0,041	0,783	-0,059

Value $\alpha_{100}(3.6)$ coincides with the corresponding value in Table 1.

Conclusions

In this paper various methods of constructing models of a regional field are considered. The transition from regional ASHA-coefficients to global coefficients is considered. For the first time an analytical transition between the coefficients of the global and regional gravity field, constructed by the ASHA method, was developed. It is shown that for such transition it is necessary to calculate integrals of the type of Fredholm integrals of the first kind.

Such transition enables the new data to be refined with existing global models using regional models. It is important not only for the construction of gravitational or magnetic fields, but also in any other cases where it is necessary to model the potential field in a limited region.

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Relationship between global and regional gravitational field

B.B. Dzhuman

With the emergence of a large number of new data, there is a need to build new and refine the existing models of gravity fields. If such data are known in a certain region, it is advisable to construct a regional gravity model and then to clarify the global model with it using, since building a new global model of high resolution requires much more time and material resources. In this paper, the method of adjusted spherical harmonic analysis (ASHA) for constructing regional gravitational fields is considered. This method involves the transformation of the initial data from the sphere cap into the hemisphere. In this case the associated Legendre spherical functions form two systems of orthogonal

functions, but in general they are not orthogonal. The analytical transition between the ASHA-coefficients of the regional field and coefficients of the global field is found. A method is suggested for finding the coefficients of transition between global and regional coefficients of the corresponding models as well as for the example found the such coefficients for a partial case. In this paper for the first time an analytical transition between a global and a regional gravitational field is found. Such transition makes it possible to compute the coefficients of the global gravitational field using exclusively regional data.