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EVALUATION OF THREE-DIMENSIONAL DEFORMATION FIELDS OF THE EARTH BY METHODS OF THE PROJECTIVE DIFFERENTIAL GEOMETRY. DILATATION FIELDS OF THE EARTH

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Statement of the problem

Research that presented in this article is aimed at the disclosure the basis of the analysis of three-dimensional deformation fields by methods of the projective differential geometry according the data of Earth's surface monitoring by GNSS.

The purpose and direction of research correspond to resolutions of the International Association of Geodesy in framework of the activities of Sub-Commission 3.2 “Crystal Deformation” of Commission 3 “Earth Rotation and Geodynamics”. General objectives of the Sub-Commission will include, in particular, “to study crystal deformation in all scales, from plate tectonics to local deformation; to contribute reference frame related work in order to better understand deformations, and to improve global, regional and local reference frames and their dynamical modeling; … to promote, develop and coordinate international programs related to observations, analysis and data interpretation for the fields of investigation mentioned above” [25].

Analysis of the research and unresolved parts of the general problem

Most current researches of problem are reduced to a numerical expression of movements and deformations by various parameters and placed in frameworks of global tectonic (kinematical) models of the Earth. The theory of plate tectonics is put as a basis for the creation of models. This is the newest mobilistic tectonic concept the reciprocal motions of lithosphere plates as absolutely rigid spherical segments of lithosphere under condition of constant radius of the Earth. Depending on the origin of input data that used at creation of models, they are divided into two types - geological and geodesics.

As a classic examples of the first type are the models NUVEL-1 [19] and NUVEL-1A [20]. More detailed and accurate geological models are PB2002 [18], MORVEL [21] and NNR-MORVEL56 [16]. In addition to delineation the geometric forms and geo-reference of lithosphere plates, models defines the relative parameters of their horizontal and angular velocities of a motion and rotation, and linear velocities, which are expressed in the Cartesian coordinate system. Based on the data from comprehensive geological and geophysical monitoring of the Earth, from chronological point of view, such parameters have a long-term nature and are expressed by the measures of geological time scale. In addition to these worthy attention another kind of geological models – dilatation model [5, 6]. The model is based on the original concept of Earth dilatation fields, which is confirmed by the same geological and geophysical data. The model is aimed at deepening the cognition of evolution of the Earth as a whole and in context the interaction of lithosphere plates. Provided confirmation of dilatation processes by various parameters from the perspective of deformation analysis this model would be harmoniously complemented the officially recognized geological models. These parameters are able to provide the geodetic monitoring of the Earth. In general, if for creation of the first geological models the usage of geodetic data was rather limited, then for such models as MORVEL and NNR-MORVEL56, a full range of results from the remote monitoring of the Earth by the methods of satellite geodesy that were accrued at the moment of models verification was used. As a consequence, part of the latest models is a set of velocities GEODVEL [17]. Wherefore the latter models are often referred to as geological-geodesics.

Numerical characteristics of motions of the Earth's surface, which are determined by the methods of satellite geodesy almost for the past 25 years, allows submit the tendencies of reciprocal motions of lithosphere plates more accurately than global geological models. In addition, they can transmit and predict current short-term plate motions as well as define their internal laws of deformation, which is caused by regional and local tectonic processes. The last factor violates the hypothesis of the absolute rigidity of major lithosphere plates; however, it reveals objective prospects for usage of geodetic methods data for allocating micro-plates with relative motion parameters of higher numerical order or anomalous features of their spatial distribution. Such prospects are implemented in the second type of global tectonic models of the Earth – geodesics. They express the absolute parameters of plate’s movements in the Earth's reference frame ITRS.

As the results of observations by methods of satellite geodesy in the ITRS readout system accrued and became consistent with the then existing geological models, empirical models of plate motions of the exclusively geodetic origin began to arise. These include, for example, the models REVEL [30], GSRM-1 [27] and its updated version GSRM v.2.1 [26], the already mentioned GEODVEL [17] and ITRF2008-PMM [14].

The latest model par with GSRM v.2.1 provides by far the most accurately expresses the absolute parameters of motions of fixed plates. For example, the estimated computation accuracy of ITRF2008-PMM is 0.3 mm/year, while for geological-geodetic model MORVEL it equals 0.67 mm/year. Such indicators are achieved mainly by raising the accuracy of Earth monitoring results by GNSS.

Deformation analysis is largely contributes to deepening a cognition of modern tectonic processes. However, this mathematical tool has found a full application only in a GSRM type model. Here is used an approach, which is based on hypothesis of sphericity of the Earth. This provides a concordance with the concept of reciprocal tectonic movements of plates as rigid spherical segments of lithosphere and justifies the use of stations coordinates in local spherical system λ, φ, r as input data. They obtained by coordinates transformations from ITRS. The essence of used methods is disclosed, for example, in articles [28, 29]. Authors actually calculate the horizontal surface deformations that attributed to geosphere and deliberately ignore vertical movements of the Earth. But as indisputable positive of solution, considers the evaluation of rotation vector of the local surface around conditional pole, which fixes the r coordinate, and associates him with a rigid rotation of the Earth around the Euler vector. According to this circumstance calculated estimates are identified with spatial deformations.

In other tectonic models deformation analysis tools are not used, although his necessity indirectly confirmed even at research articles that were cited. Thus, if we analyze the results of motion study's only within the Eurasian plate, then, for example, according to [14], the north-western part of the European continent close to the Scandinavian peninsula has its own, different from the rest of the territory, motion and deformation laws. Other tendencies of surface deformations are observed in the Mediterranean basin [21]. Otherwise, according to the hypothesis [17], the Eurasian plate must be divided into two independent parts along the Ural Mountains. Evaluation of movements only by velocities of GNSS-stations linear displacements or horizontal angular velocities of rotation is insufficient for official confirmation these hypotheses and the division of the Eurasian plate on micro-plates. As a telling example in this regard, is the result of direct studies that presented in the article [15]: by using the tools of deformation analysis, the authors identified in the Mediterranean basin within the limits of Eurasian the independent Adriatic micro-plate.

My own vision of the current state the use of a deformation analysis for the needs of geodynamics presented in article [12]. The vast majority of researches aimed at evaluating of the horizontal component of deformation fields. This approach has become a heritage of mass use the results of repeated observations in classical geodetic planned networks as recently the only available inputs data to solving the problem. These data for today practically already have lost their relevance. Instead of it, new much broader prospects have data the GNSS-monitoring of stations coordinates in ITRS. But this causes the rethinking of traditional approaches to the deformation analysis due to the need to create models that are adequate to ITRS from the standpoint of ability to fully evaluate the three-dimensional deformation fields. Solving the problem in this context is proposed to perform from the standpoint of the theory of differential presentation the transformations (mappings) of space images and use the projective differential (metric) geometry methods [1, 13]. According to the hypothesis that mapping has a geophysical origin, it is identified with the deformation of the Earth as a domain of space. Then, using the three-dimensional metric tensor of space and various measures of the deformation reveals perspectives to describe the phenomenon by different in content numerical characteristics. Taking into account the established practice of deformation analysis, these characteristics are divided into three groups: 1) main linear deformations - parameters of the form change in the specified direction; 2) angular distortion parameters; 3) dilatation - parameters of relative changes in the volume of the Earth or area of its surface while preserving the overall form.

Formulation of the problem

Based on the theory of differential presentation the transformations of space images, in this part of the researches will focus on evaluation of Earths dilatation fields in different scales and search of solutions by the input data in the ITRS. Let us try to present the results of solutions in such general form that would ensure the expression of nonlinear dilatation patterns.

The main material of researches

Let the closed and continuous domain of transformation (mapping) Δ is the Earth as a spatial body of planetary scale and $x_i = X_i^1$, $y_i = X_i^2$, $z_i = X_i^3$ coordinates of points M_i ($i = \overrightarrow{1, N}$) meet the conditions of the Earth parameterization by the ITRS. Points M_i are the GNSS-stations which are located on its physical surface. Designations $\left(X^1, X^2, X^3\right)$ are identical to (x, y, z) and introduced solely for the purpose of a compact presentation of the following intermediate and final results of the problem solution. If coordinates $x_i' = X_i'^1$, $y_i' = X_i'^2$, $z_i' = X_i'^3$ defines the positions of points M_i', which due to deformation are the mapping of points M_i, defines a transformed domain Δ' and Δ' remained the closed and continuous, than the mapping Δ to Δ' always can be expressed analytically by equations

$$\left.\begin{aligned} X'^1 &= u(X^1, X^2, X^3) \\ X'^2 &= v(X^1, X^2, X^3) \\ X'^3 &= w(X^1, X^2, X^3) \end{aligned}\right\} . \qquad (1)$$

The general theory of mappings imposes on the base functions of transformation (1) the homeomorphism conditions: the uniqueness, continuity and differentiability (Jacobian different from zero). But do not limit their analytical forms. This allows you to describe the transformation by any smooth or piecewise smooth functions which can be determined by the displacements $X_i'^k - X_i^k$ ($k = \overrightarrow{1,3}$) along a certain way parametrically given curve. In terms of the formulated objectives, this provides the prospect of the transfer within the functional model (1) the nonlinear transformations. In addition, it is also important that coordinates X_i^k can be set in an arbitrary three-dimensional system.

Previous stages of researches have shown the following: in formulated statement the solution is achieved based on the properties of transformations the Riemannian space R_n in the form of its elementary or complicated diffeomorphic manifolds – a couple of non-isotropic manifolds of the same dimension n, that are subject to the mutually unambiguous and continuously differentiated mapping of the $n-1$ class. Such transformations respectively define the class of smooth or piecewise smooth functions that are able to transmit these transformations, i.e. C^{n-1}. As used mathematical tools are methods of the projective differential geometry and methods of description the changes of diffeomorphic manifolds metrics. Such decision is grounded on fundamentals of tensor analysis in the Riemannian space environment [7].

Considering the task statement, as complicated manifold is expedient to use the Euclidean space E_3 that is tangent to each point of the Riemannian space R_3 in the form of local three-dimensional coordinate frames in the Cartesian system. Then the tensor field with the Riemannian metric, which defines the functional model (1), transforms into a geometric image of tensors in tangent Euclidean space with the corresponding to it metric. Thus, for solving the problem need only withstand the conditions of sufficiently smooth changes of Riemannian metric in the transition from point to point. These conditions must ensure the adequately constructed model (1) by the continuous differentiability of its base functions.

The coordinate system X^k ($k = \overrightarrow{1,3}$), in which is expressed the position of points M_i in Δ domain, is a rectangular Cartesian and identified with the ITRS. In such system the metrics of space defines an invariant differential quadratic form (designation of the sum by Einstein rule)

$$ds^2 = \delta_{ij} dX^i dX^j . \qquad (2)$$

This form identifies the linear element of the Δ domain. Since the coordinate axes are orthogonal, the metric coefficients δ_{ij} are such that

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

The metric form of the deformed domain Δ' is associated with the linear element ds', that is a mapping of ds:

$$ds'^2 = e_{ij} dX^i dX^j . \qquad (3)$$

Metric coefficients e_{ij} of the quadratic form (3) generate a symmetric matrix which is called the main bivalent metric tensor the transformation (deformation) of the space domain:

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} . \qquad (4)$$

The algorithm of disclosure the coefficients is the following:

$$e_{11} = \left(\frac{\partial u}{\partial X^1}\right)^2 + \left(\frac{\partial v}{\partial X^1}\right)^2 + \left(\frac{\partial w}{\partial X^1}\right)^2 ;$$

$$e_{22} = \left(\frac{\partial u}{\partial X^2}\right)^2 + \left(\frac{\partial v}{\partial X^2}\right)^2 + \left(\frac{\partial w}{\partial X^2}\right)^2 ;$$

$$e_{33} = \left(\frac{\partial u}{\partial X^3}\right)^2 + \left(\frac{\partial v}{\partial X^3}\right)^2 + \left(\frac{\partial w}{\partial X^3}\right)^2 ;$$

$$e_{12} = \frac{\partial u}{\partial X^1}\frac{\partial u}{\partial X^2} + \frac{\partial v}{\partial X^1}\frac{\partial v}{\partial X^2} + \frac{\partial w}{\partial X^1}\frac{\partial w}{\partial X^2} ;$$

$$e_{23} = \frac{\partial u}{\partial X^2}\frac{\partial u}{\partial X^3} + \frac{\partial v}{\partial X^2}\frac{\partial v}{\partial X^3} + \frac{\partial w}{\partial X^2}\frac{\partial w}{\partial X^3} ;$$

$$e_{13} = \frac{\partial u}{\partial X^1}\frac{\partial u}{\partial X^3} + \frac{\partial v}{\partial X^1}\frac{\partial v}{\partial X^3} + \frac{\partial w}{\partial X^1}\frac{\partial w}{\partial X^3} . \qquad (5)$$

The volume of any closed continuous domain of space always defines a determinant of tensor, which is composed by metric coefficients in the coordinate system of this space [2, 8]. Accordingly, the element dV of volume for the Δ domain $dV = \sqrt{\delta} dX^1 dX^2 dX^3$. But a determinant of the quadratic form (2) $\delta = \det \delta_{ij} = 1$, therefore $dV = dX^1 dX^2 dX^3$. The element dV' of volume for the Δ' domain $dV' = \sqrt{e} dX^1 dX^2 dX^3$, where $e = \det e_{ij}$ is a determinant of the quadratic form (3). Ratio

$$\frac{dV'}{dV} = \sqrt{e} \qquad (6)$$

expresses the change in the volume of space domain. The value $\theta_{абс} = \sqrt{e}$ is called the absolute coefficient of volume expansion. As a determinant of e_{ij} tensor, (6) is an absolute invariant. In the expanded form

$$\frac{dV'^2}{dV^2} = \theta^2_{абс} = e_{11}e_{22}e_{33} + 2e_{12}e_{13}e_{23} -$$
$$- e_{11}e^2_{23} - e_{22}e^2_{13} - e_{33}e^2_{12} \,. \quad (7)$$

Based on the invariant e, we can construct other invariants of e_{ij} tensor. Considering the measure of deformation

$$ds'^2 - ds^2 = \left(e_{ij} - \delta_{ij}\right)dX^i dX^j \,, \quad (8)$$

if subtracts from e_{ij} a tensor with constant coefficients $\lambda\delta_{ij}$ (λ – an arbitrary constant), we obtain the invariant of transformation $\det\left(e_{ij} - \lambda\delta_{ij}\right)$. When we expand the determinant and pairs the members with same degrees relatively to λ, we obtain a cubic polynomial

$$\det\left(e_{ij} - \lambda\delta_{ij}\right) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 \,, \quad (9)$$

where invariants in the expanded form

$$I_1 = e_{11} + e_{22} + e_{33} \,, \quad (10)$$

$$I_2 = e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33} - e^2_{12} - e^2_{13} - e^2_{23} \,, \quad (11)$$

$$I_3 = e_{11}e_{22}e_{33} + 2e_{12}e_{13}e_{23} - e_{11}e^2_{23} - e_{22}e^2_{13} - e_{33}e^2_{12} \,. (12)$$

Note the identity of the formulas (7) and (12): invariant I_3 is associated with the absolute coefficient of volume expansion.

By analogy to (8) make uses another measure of the deformation:

$$dV'^2 - dV^2 = \det\left(e_{ij} - \delta_{ij}\right)\left(dX^1 dX^2 dX^3\right)^2 . \quad (13)$$

From the ratio $\frac{dV'^2 - dV^2}{dV^2}$ obtains the same cubic polynomial (9) with invariants (10) - (12). Thus, a determinant (9) expresses the change in volume per unit volume as a relative coefficient $\theta_{відн}$ of volumetric expansion of the Δ domain (provided the $\lambda = 1$):

$$\frac{dV'^2 - dV^2}{dV^2} = \theta^2_{відн} = e_{11} + e_{22} + e_{33} - e_{11}e_{22} -$$
$$- e_{11}e_{33} - e_{22}e_{33} + e^2_{12} + e^2_{13} + e^2_{23} + e_{11}e_{22}e_{33} +$$
$$+ 2e_{12}e_{13}e_{23} - e_{11}e^2_{23} - e_{22}e^2_{13} - e_{33}e^2_{12} - 1 \,. \quad (14)$$

Presented above results are identical to task solutions in the theory of nonlinear deformation that are disclosed, for example, in [4, 8 and 24]. This theory considered other approaches to the expression of dilatation parameters. Is noteworthy an approach, which based on the use of deformation measure (8). The ratio

$$\frac{ds'^2 - ds^2}{ds^2} = \left(e_{ij} - \delta_{ij}\right)\frac{dX^i}{ds}\frac{dX^j}{ds} = \mu^2$$

allows to express the coefficient (or module or scale) of relative expansion μ in any direction, which sets the values $\frac{dX^k}{ds} = \cos\eta^k$ – the directional cosines of angles η^k which forms a linear element with its projections on the coordinate axes ($k = \overrightarrow{1,3}$). Solution of the equation system $\frac{d\left(\mu^2\right)}{d\eta^k} = 0$ gives an unambiguous and only possible triad of directions that preserve the orthogonality after deformation, indicating the extreme deformations of space. Extreme deformations are expressed by coefficients of relative elongations μ^k_{ext}; they belong to the group of main linear deformations. The product of coefficients μ^k_{ext} defines the absolute and relative coefficients of volumetric expansion in the forms (7) and (14).

Directions of extreme elongations can be set in the geocentric polar system relatively to planes of equator X^1OX^2 and zero meridian X^1OX^3. Considering the established practice of using the polar coordinates in geodesy, the orientation of directions by geocentric latitude φ and longitude λ seems more expedient compared with the same by angles η^k. Replacement the directional cosines by values $\frac{dX^1}{ds} = \cos\varphi\cos\lambda$, $\frac{dX^2}{ds} = \cos\varphi\sin\lambda$ and $\frac{dX^3}{ds} = \sin\varphi$ gives for the triad of main orthogonal directions $(\lambda_\circ + 90^\circ, \varphi_\circ, \varphi_\circ + 90^\circ)$ the corresponding coefficients of extreme elongations $\left(\mu^2_{12\min}, \mu^2_{\max}, \mu^2_{\min}\right)$ ($\mu^2_{12\min}$ expresses a minimum elongation on the equatorial plane in a direction $\lambda_\circ + 90^\circ$):

$$\mu^2_{12\min} = \frac{1}{2}\left(e_{11} + e_{22} - \gamma\right);$$

$$\mu^2_{\max} = \frac{1}{4}\left(e_{11} + e_{22} + 2e_{33} + \gamma + \sqrt{\left(e_{11} + e_{22} - 2e_{33} + \gamma\right)^2 + \frac{8}{\gamma}\left(e^2_{13}\left(\gamma + e_{11} - e_{22}\right) + e^2_{23}\left(\gamma + e_{22} - e_{11}\right) + 8e_{12}e_{13}e_{23}\right)}\right);$$

$$\mu^2_{\min} = \frac{1}{4}\left(e_{11} + e_{22} + 2e_{33} + \gamma - \sqrt{\left(e_{11} + e_{22} - 2e_{33} + \gamma\right)^2 + \frac{8}{\gamma}\left(e^2_{13}\left(\gamma + e_{11} - e_{22}\right) + e^2_{23}\left(\gamma + e_{22} - e_{11}\right) + 8e_{12}e_{13}e_{23}\right)}\right),$$

where $\gamma = \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}$ is a maximum shear in equatorial plane. The product of these coefficients makes it possible to obtain formulas for expression of absolute and relative indicators of dilatation in such forms:

$$\theta_{абс}^2 = e_{11}e_{22}e_{33} - e_{33}e_{12}^2 + \\ + \frac{1}{2\gamma}\Big(2e_{12}e_{13}e_{23}(\gamma - e_{11} - e_{22}) - e_{11}e_{23}^2(\gamma - e_{11}) - \\ - e_{22}e_{13}^2(\gamma - e_{22}) + \left(e_{13}^2 + e_{23}^2\right)\left(2e_{12}^2 - e_{11}e_{22}\right)\Big); \quad (15)$$

$$\theta_{відн}^2 = e_{11} + e_{22} + e_{33} - e_{11}e_{22} - e_{11}e_{33} - e_{22}e_{33} + \\ + e_{12}^2 + e_{11}e_{22}e_{33} - e_{33}e_{12}^2 + \frac{1}{2\gamma}\big(2e_{12}e_{13}e_{23}(\gamma + 2 - \\ - e_{11} - e_{22}) - e_{11}e_{23}^2(\gamma + 1 - e_{11}) - e_{22}e_{13}^2(\gamma + 1 - \\ - e_{22}) + e_{13}^2\left(\gamma + e_{11}(1 - e_{22}) + 2e_{12}^2\right) + \\ + e_{23}^2\left(\gamma + e_{22}(1 - e_{11}) + 2e_{12}^2\right)\big) - 1. \quad (16)$$

Presented results designed to express the final deformations of any nonlinear nature, which is able to transmit the homeomorphic functional model. In the case when describes a small deformations, in all formulas enough to leave only quantities of the first order of smallness. Such approximation is a practically permissible from the standpoint of the simplest linear form of elasticity theory, which studies the infinitesimal centralized-affine transformations of spatial bodies. This kind of transformations are considered in metrics of the affine space, which is the trivial not only relatively to a Riemannian space, but also even Euclidean. As a carrier of information about domain deformations in such space are taking into account a bivalent tensor, which is called the affinor. He is the consequence of a linear coordinate transformation with the appropriate matrix which coincides with a matrix of affinor components. For such transformation a determinant is disclosed neglecting the small quantities of second and higher orders, but as a consequence it is only a trace of the affinor. Then the determinant as a numeric indicator the change in the volume of any body, is identified with the invariant (10): $I_1 = e_{11} + e_{22} + e_{33}$. Such argumentations gleaned from [7]. When evaluates the dilatation in a practice of deformation analysis for needs of geodynamics, uses a same simplified formula of the form (10). But she is only a consequence of "local linear approximation of the actual nonlinear deformation in any point" [22]. In this regard, as a fact states the following: the approximate expression of dilatation by invariant I_1 is unacceptable except for the case when a linear character of deformation is a priori confirmed. Attempts to use the model (1) with non-linear base functions or next their linearization leads only to complications of the problem, but does not provide the expected from such model adequate effect. In fact, at the deterministic relation "function-tensor" based on the classical linear theory of elasticity takes part not the function that expresses the deformation, but only its local linear approximation in infinitely small scale. Exactly such approximation defines the structure of tensor (affinor) and associated invariants of the linear deformation. Overall, the expression of nonlinear deformations in the part of dilatation evaluation able to provide the result that is presented above.

If consider the obtained results from the standpoint of the area of their practical application, the conclusion is an unequivocal – evaluation of the global dilatation field of the Earth, attributed not to one or another modeled, but to the actual topographic surface. However, the definition of effective, reliable and unshifted estimates of the deformation burdened by the condition of sufficient GNSS-stations covering a surface of the Earth, as a spatial body, at least on the scale of hemisphere. This condition creates the problem that is associated with the definition of coordinates within the oceans. Theoretically, this problem could be solved by a global grid construction, for example, in the form of spherical or ellipsoidal quadrangles or triangles, if involve the Delaunay triangulation method, followed by extrapolation of displacements in grid nodes relative to stations in mainland and insular parts of the Earth. Such a mathematical tool used in the research practice (see, e.g., [Marchenko et al., 2012]). But the accuracy of extrapolation to great distances will be low and for this reason will probably lose its sense the idea of expressing the nonlinear deformation on a planetary scale. In this regard, given problem requires a comprehensive discussion and detailed study and for today remains open.

Presented above results were obtained under the condition that the transformation domain is parameterized by the Cartesian rectangular coordinate system. Such a parameterization is caused by the origin of input survey data, which obtained by the GNSS method in ITRS system as a particular case of rectangular Cartesian. Considering the invariance of the basic quadratic form of space concerning the choice of coordinate systems in diffeomorphic manifolds, the identical result of solution can be achieved in any other three-dimensional system, even in a curvilinear, for example, in traditional for a geodesy systems such as spherical or ellipsoidal. It should be noted, that from the viewpoint of Riemannian geometry of this type three-dimensional parameterizations, be equally as Cartesian in the tangent Euclidean space, are used for manifolds categories which are called as complicated. For an example, we can take a spherical coordinate system. Taking into account that the position of a point in space is unambiguous irrespective of the coordinate system, wherewith he parameterized, accordingly must be unambiguous the relationship between different systems. For rectangular Cartesian and spherical systems such relationship expresses the proverbial formulas

$$X^1 = x^1 \sin x^2 \cos x^3,$$
$$X^2 = x^1 \sin x^2 \sin x^3,$$
$$X^3 = x^1 \cos x^2, \qquad (17)$$

where x^k are spherical coordinates of the point. A linear element of space in Cartesian coordinates expresses the formula (2). Since $dX^i = \frac{\partial X^i}{\partial x^\alpha} dx^\alpha$ ($\alpha = \overrightarrow{1,3}$), then in spherical coordinates $ds^2 = g_{ij} dx^i dx^j$, where $g_{ij} = \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\alpha}{\partial x^j}$ is a covariant symmetric tensor of transformation in the given point. In coefficients of this tensor the relationships (17) are included.

The last example gives grounds to state the following. In the context a modeling of global dilatation field of the Earth no need to examine a complicated manifold with any other parameterization except the Cartesian. Coordinates of GNSS-stations, as input data of a model, expressed exactly in this system and the transformation of the type (17) only complicates the problem solving. The coordinate system in which is parameterized a manifold, by no means will not affects on estimates of the dilatation field, since the transformation of systems is laid down in the structure of tensor at the point and of a tensor field as a whole. An exception can make up a case where the model (1) is based on base functions of the harmonic type, for example, on series of spherical functions which are traditionally used in the modeling of gravitational and other force fields of the Earth. Then is used a spherical coordinate system. Such exception is justified by prospect of a compatible interpretation of these fields and the deformation field of the Earth within the same functional model.

If the presented above results as a whole and the last conclusion in particular relating to the construction of a global dilatation model of the Earth, then in a completely different statement should formulate the problem when creating models of regional (e.g., for separate lithosphere plates) and, the more, of local scales. Argumentation of this assertion is a trivial. Formulas (7), (14) or (15), (16) express the coefficients of volumetric expansion of the space domain. If as such domain without regard to the Earth as a whole consider, for example, the lithosphere plate, then to determine the coefficients need to coordinate its surface entirely that a priori is impossible. Thus, the problem of determining the coefficients of volumetric expansion for a separate plate is devoid of logical content. Measurements by a GNSS method are carried out only on physical surface of the Earth, so the solution of the problem the evaluation of regional dilatation fields should be considered in the context a tangent surfaces of Riemannian space. It needs the annotation of some additional theoretical formulations.

In general, any curvilinear surface, including a plane as surface of zero curvature, is a R_m manifold of R_n space. The elementary m-dimensional surface in n-dimensional manifold it is a plural of points which are defined by parametric equations $X^i = X^i\left(u^1, \ldots, u^m\right)$. $i = \overrightarrow{1, N}$; $u^1, \ldots, u^m$ are independent parameters of the surface (coordinates) which vary in the Δ_u domain of R_m manifold. It is always, $m = \overrightarrow{1, n-1}$. If the elementary manifold R_m is parameterized by the coordinate system $u^1, \ldots, u^m$, then in the form of continuously differentiated transformations of the $m-1$ class is permitted to replace it to others diffeomorphic manifold and the Δ_u domain can transform into the corresponding to it Δ'_u domain. This mapping can transmit a smooth or piecewise smooth functions of the C^{m-1} class. During mapping any point of the Δ_u domain is undergoing an infinitesimal displacement along the ds arc, therefore $ds^2 = g_{ij} dX^i dX^j$, where g_{ij} is a tensor of mapping. Taking into account that $dX^i = \frac{\partial X^i}{\partial u^\alpha} du^\alpha$ and $dX^j = \frac{\partial X^j}{\partial u^\beta} du^\beta$, then differential of the arc $ds^2 = G_{\alpha\beta} du^\alpha du^\beta$, where $G_{\alpha\beta} = g_{ij} \frac{\partial X^i}{\partial u^\alpha} \frac{\partial X^j}{\partial u^\beta}$. $G_{\alpha\beta}$ is the metric tensor of surface, unless the $\det G_{\alpha\beta} \neq 0$ and $G_{\alpha\beta} = G_{\beta\alpha}$, i.e., the surface is non-isotropic. In coefficients of the $G_{\alpha\beta}$ tensor are already included the transformation of curvilinear coordinates $u^1, \ldots, u^m$ of the R_m manifold into spatial coordinates of the Riemannian space R_n. Thus, the R_m surface in the R_n space itself is a Riemannian space [2, 7].

Considering the following practical application of presented theoretical formulations should be noted that not any curvilinear surfaces can be regarded as an elementary. For example, a sphere or equally ellipsoid a whole, as models of the Earth, are complicated three-dimensional manifolds. They are subject to description as a tangent three-dimensional space in R_3 with the corresponding consequences, which were discussed earlier with the aim of modeling the global deformation fields. “As elementary manifold can be considered only the surface of a hemisphere (or "half-ellipsoid"), but not including the equatorial section” [7]. The latter circumstance is a motivation to the application of these theoretical formulations for modeling the deformation fields of regional scale.

The simplest example of implementation the modeling of this type gives the theory of surfaces in an ordinary Euclidean space E_3 [1]. On the surface that is parameterized by the curvilinear coordinate system, arises the first basic quadratic form which expresses the square of the arch differential by the formula

$$ds^2 = E(u,v)du^2 + 2F(u,v)dudv + H(u,v)dv^2. \quad (18)$$

Thus, the surface can be considered as a two-dimensional Riemannian space with the metric quadratic form (18) and the metric tensor $G_{11} = E(u,v)$, $G_{12} = G_{21} = F(u,v)$, $G_{22} = H(u,v)$. Riemannian geometry, which is generated on the surface in the tangent Euclidean space by quadratic form (18), is called the internal geometry of the surface [1]. According to the authors terminology in article [23], such a surface is called "embedded in Euclidean space" and a means of constructing the metric tensor - "internal modeling".

Using the latest theoretical formulation can evaluate the regional or local deformation fields of the Earth only in their horizontal components. It causes the description of dilatation fields by absolute or relative indicators of changes in the area for domains, which belongs to diffeomorphic two-dimensional manifolds. Such a theoretical basis we used in earlier studies. This made it possible to achieve the solutions in typical geodetic coordinate systems that are traditionally used for parameterization of surfaces in the framework of different models of the Earth. So, in the article [11] presented the results of solutions in the spherical geocentric system, in [9] - in the geodetic (ellipsoidal). For the deformations evaluation of the local scale enough use the results of the problem solution on a plane in rectangular system, for example, in the Gauss-Kruger projection. Such results are contained in the article [10]. Although these solutions obtained independently, but do not qualify for the antecedence and scientific novelty because similar to them has previously been used in the research practice. For example, the authors of the article [29] and other followers of this direction in their studies used a local spherical coordinate system. Studies [23] and others of the same type to them are based on the use of ellipsoidal system. However, its own results differ middle of mentioned by the fact that submitted in the generalizing unified form which is able to express the deformations of any character, while others mentioned at forming the tensor include a linearization of base functions of coordinates u, v. From this perspective, its own results have the unquestionable practical importance for geodynamic studies.

An important remark: “if the surface is parameterized by one or another coordinate system, then corresponding to its coordinate lines is completely belongs to this surface” [2]. Thus, the metric tensor and deformation estimates that follow from him will belongs solely to this surface. If such a circumstance regarded in the context of our objectives, then we must to take into account that in a internal modeling the deformation estimates will refer not to the topographic surface, as in their studies say the authors of the article [23], but to a model surface of the Earth just such in the coordinate system of which are defined the input geodetic data. This usually is a geosphere or the Earths ellipsoid of revolution. Implement a parameterization of the topographic surface is possible only within a complicated Riemannian manifold in the form of tangent three-dimensional Euclidean space, as was presented above. For today, as elementary two-dimensional manifold can not be used even such a best approximation of topographic surface as geoid because its surface is not subject to parameterization by any traditional geodetic coordinate system.

Conclusions

In the presented part of studies we have outlined the perspectives of using the data of GNSS-monitoring of a physical surface of the Earth for the evaluation of its dilatation fields in different scales. Problem solution achieved by methods of the projective-differential (metric) geometry based on the theory of differential representation the transformations of Riemannian space images in the form of its diffeomorphic manifolds. On this basis, prospects for expression of deformation fields by nonlinear functional models are substantiated; were obtained the following results.

1. On condition the parameterization of complicated Riemannian manifolds by Cartesian system in a tangent three-dimensional Euclidean space derived analytical expressions for the coefficients of absolute and relative volumetric expansions that express the dilatation of the Earth in a global scale. In this formulation evaluation of the global dilatation field can be achieved by a direct using of spatial geocentric coordinates of GNSS-stations as the input data for solving the problem. Estimates refer to the topographic surface of the Earth.

2. Expressions for estimates of dilatation fields of the Earth in regional and local scales are substantiated. Their transfers the coefficients of changes in the area of a being evaluated Earth's surface. Such result is achieved within elementary Riemannian manifolds that are parameterized by curvilinear coordinate systems on the tangents two-dimensional surfaces. Then the evaluation of dilatation fields carried out by a priori transformed spatial geocentric coordinates into the curvilinear (spherical or ellipsoidal) systems or by the direct use of spatial coordinates. In the latter case, the coordinate transformation algorithm is incorporated into the structure of the metric tensor as a geometric image of manifold in the Riemannian space. Estimates of dilatation fields in regional and local scales refer exclusively to model curvilinear surfaces.

Provided the proper coverage of the Earth by GNSS-stations and adequately constructed a functional model of transformations their coordinates, obtained results of solutions able to provide an effective reliable estimates of dilatation fields of the Earth. Considering also the ability to transfer non-linear effects of deformation processes, the informative potential of proposed solutions is significantly higher as compared to traditionally used. The proposed methods of modeling may become an additional means of knowledge the evolution of many geodynamic phenomena within existing tectonic models or, moreover, a mentioned earlier dilatation model of the Earth. The dilatation description by various estimates in any of their scale is able to confirm the dilatation concept [5, 6] or, equally, to put it into doubt, if phenomenon will not get a sufficient numerical confirmation.

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In this article presents a result of solving the problem an evaluation of three-dimensional deformation fields of the Earth in different scales in the part of expression the dilatation. Solutions achieved by methods of the projective differential geometry based on a theory of the differential representation of a space images mapping.