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USING THE TOROIDAL SURFACES FOR DEVELOPING OF MATHEMATICAL BASES OF GEOGRAPHICAL MAPS OF UKRAINE

P. Korol, V. Voloshin
Lesya Ukrainka Eastern European National University

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Statement of a problem

In general, for mathematical cartography the idea of using intermediate surfaces of revolution for transition from the surface of the sphere to the plane of the map is not new.

In 1943, the famous Hungarian cartographer and geographer, curator of the Harvard Map Collection Erwin Josephus Raisz (1893-1968) proposed a series of map projections, construction of which involves the use of intermediate curved surfaces [3]. For the definition of such type projections author was used term orthoapsidal (from Greek *ορθο* – orthographic + *αψίδα* – apsidal), which defines the geometric principle of their construction: topologically, the spherical surface is converted onto some other apsidal surface of revolution, which in turn is orthogonally projected on the plane of the map. Except of traditionally accepted in mathematical cartography circular cylindrical and conic, to the class of apsidal surfaces of revolution, on which the sphere is projected, belongs the ellipsoid of revolution, elliptic paraboloid, hyperboloid of two sheets, torus etc. In [3] Raisz suggested the various options for using apsidal surfaces of revolution for developing of mathematical bases of world maps, in particular a surface of ellipsoid of revolution with different inclination angles and eccentricity, and also surfaces of hyperboloid and torus, that the projected on the plane of the map create the illusion of volumetric images and provide the effect of spherical Earth.

Relations to important scientific and practical tasks

Scientists of Lesya Ukrainka Eastern European National University performs the system applied researches of modifications of the existing map projections, that meet the requirements of the creation, preparation and publication geographical maps of Ukraine.

A purpose of this research is to study the possibilities of using a modified version of Raisz arbitrary isometric orthoapsidal projection at the conclusion of different scale maps of Ukraine.

Analysis of the latest researches and publications relating to resolve this problem

The historical aspects of the development of orthoapsidal map projections highlighted in the monograph J. P. Snyder [6], and their mathematical justification at spherical variant made in paper E. J. Raisz [3]. The possibilities of using apsidal surfaces of revolution, including torus, for developing of mathematical bases of world maps contained in textbooks E. J. Raisz [2, 4].

Mathematical justification of Raisz modified isometric orthographic projection on the half of torus Armadillo, practical aspects of constructing its mapping grid, features of constructing of the coordinate system, character the distribution of distortions in it and layouts of mapping grids are in the book J. P. Snyder and Ph. M. Voxland [5], and its full analytical research presented in monograph E. W. Grafarend and F. W. Krumm [1]. The possibilities of using a modified version of the projection for developing the mathematical bases of geographical maps of Ukraine in domestic and foreign cartographic literature is not considered.

Basic material

The most famous orthoapsidal map projection is Raisz modified isometric orthographic projection on half surface of torus, which called Armadillo, because the exterior of its mapping grid remotely resembling the armor of this chordate mammal.

Torus is a surface of revolution generated by revolving a circle in three-dimensional space about an axis coplanar with the circle, but not passing through its center. In this case the rotation axis of the torus can cross the circle, to touch it or be placed outside. In the first two cases, a torus called closed, in the latter – open or ring. In the open torus the annular surface of rotation called *toroidal*.

The parallels and meridians on the outer surface of the torus is equidistant circular arcs, which when projection on the plane of map are converted into non-equidistant elliptical arcs.

Raisz projection Armadillo with zero inclination of the torus looks like a pseudocylindrical projections the German geographer and cartographer Friedrich Eduard Max Eckert-Greifendorff – Eckert III and Eckert IV, but in its essence is not so.

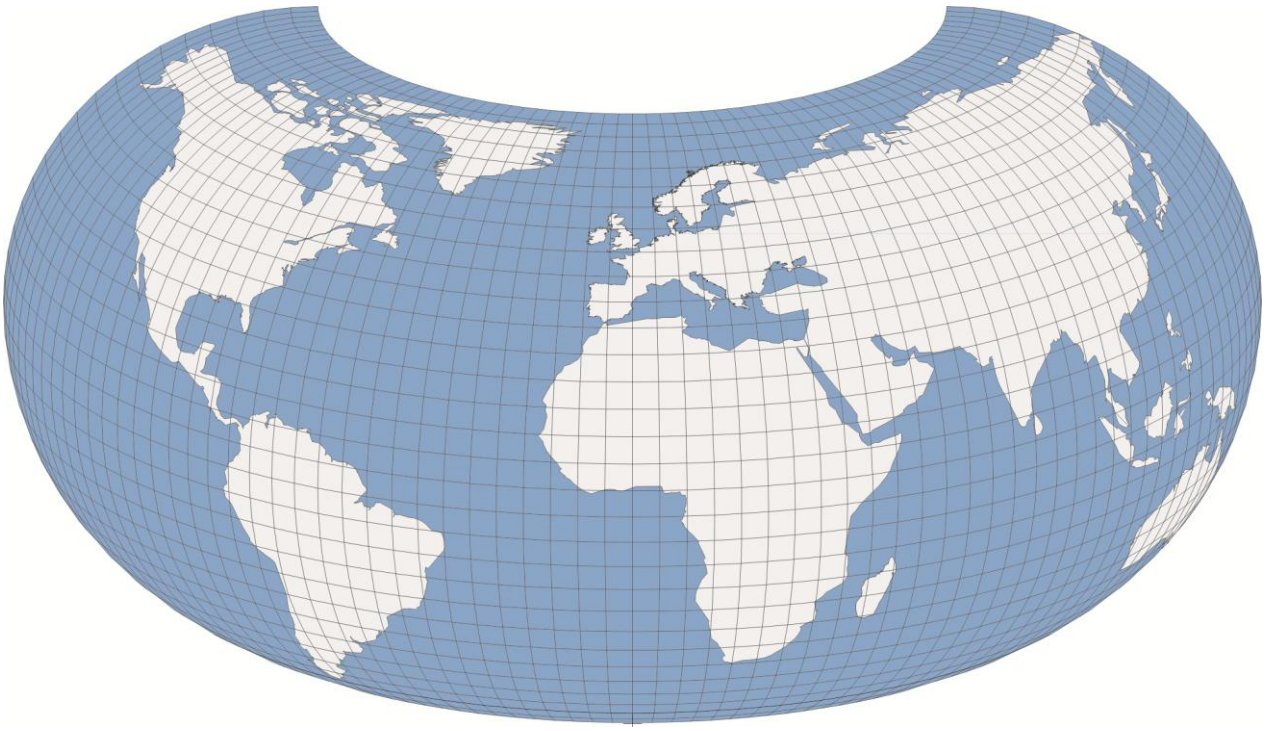

Fig.1. Raisz modified isometric orthographic projection on the half of the torus – Armadillo (inclination of the torus $\beta = 20^\circ$, central meridian $\lambda_0 = 0^\circ$)

In the original version of projection used oblique torus with inclination $\beta = 20^\circ$, which ensures the achievement of maximum disproportion in the ratio of the areas of land and ocean surface in their projection on a plane map, and the straight central meridian $\lambda_0 = 10^\circ$ E, which is the axis of symmetry (fig. 1). In this case, the North Pole is displayed as a half of ellipse, Australia shown partially, and the South Pole as and territory of Antarctica and New Zealand in the projection does not appear. For their display uses the special maps. Thus a projection artificially extends the Western and Eastern Hemispheres, covering in latitude about 410° which allows objectively divide Alaska and Siberia.

Latitude of southern parallel φ_s, which limiting the visible part of the image outer surface of the inclining torus, defined as:

$$\varphi_s = -\operatorname{arctg}\left[\cos\left((\lambda - \lambda_0)/2\right)/\operatorname{tg}\beta\right], \qquad (1)$$

where β – inclination angle of the torus.

If $\varphi \geq \varphi_s$, then Raisz oblique arbitrary orthoapsidal projection on the half of the torus in rectangular coordinates describes by system of equations at the form:

$$\begin{aligned} x &= R\left[(1+\cos\varphi)\sin\left((\lambda-\lambda_0)/2\right)\right] \\ y &= R\left[(1+\sin\beta-\cos\beta)/2+\cos\beta\sin\varphi - \right. \\ &\quad \left. \sin\beta(1+\cos\varphi)\cos\left((\lambda-\lambda_0)/2\right)\right] \end{aligned} \qquad (2)$$

Axis y coincides with the central meridian and directed to the north, and the axis x crosses the central meridian at the point with latitude $28^\circ 06'$ N and directed to the east.

Partial derivatives $\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}$ defined as:

$$\frac{\partial x}{\partial \varphi} = -R\sin\varphi\sin\left((\lambda-\lambda_0)/2\right), \qquad (3)$$

$$\frac{\partial y}{\partial \varphi} = \frac{1}{2}R\left[(1+\cos\varphi)\cos\left((\lambda-\lambda_0)/2\right)\right], \qquad (4)$$

$$\frac{\partial x}{\partial \lambda} = R\left[\cos\varphi\sin\beta+\sin\varphi\cos\beta\cos\left((\lambda-\lambda_0)/2\right)\right], \qquad (5)$$

$$\frac{\partial y}{\partial \lambda} = \frac{1}{2}R\left[(1+\cos\varphi)\cos\beta\sin\left((\lambda-\lambda_0)/2\right)\right]. \qquad (6)$$

On the basis of partial derivatives the Gauss coefficients of the first quadratic form E, F, G was calculated, the complete analytical research of projection was performed and the parameters of distortions m, n, p, a, b, ω was defined.

It should be noted, that oblique variations of this projection are very convenient to display the ratio «country in the world». Thus the inclination angle of image is determined by the average latitude of the territory. For the territory of Ukraine should be used oblique torus with inclination $\beta = 48^\circ$ and a straight central meridian $\lambda_0 = 31^\circ$ E (fig. 2, 3).

The results of analytical researches confirm the minimal distortions in orthogonality of lines of map grid of modified projection (tab. 1), which makes it as quasiconformal.

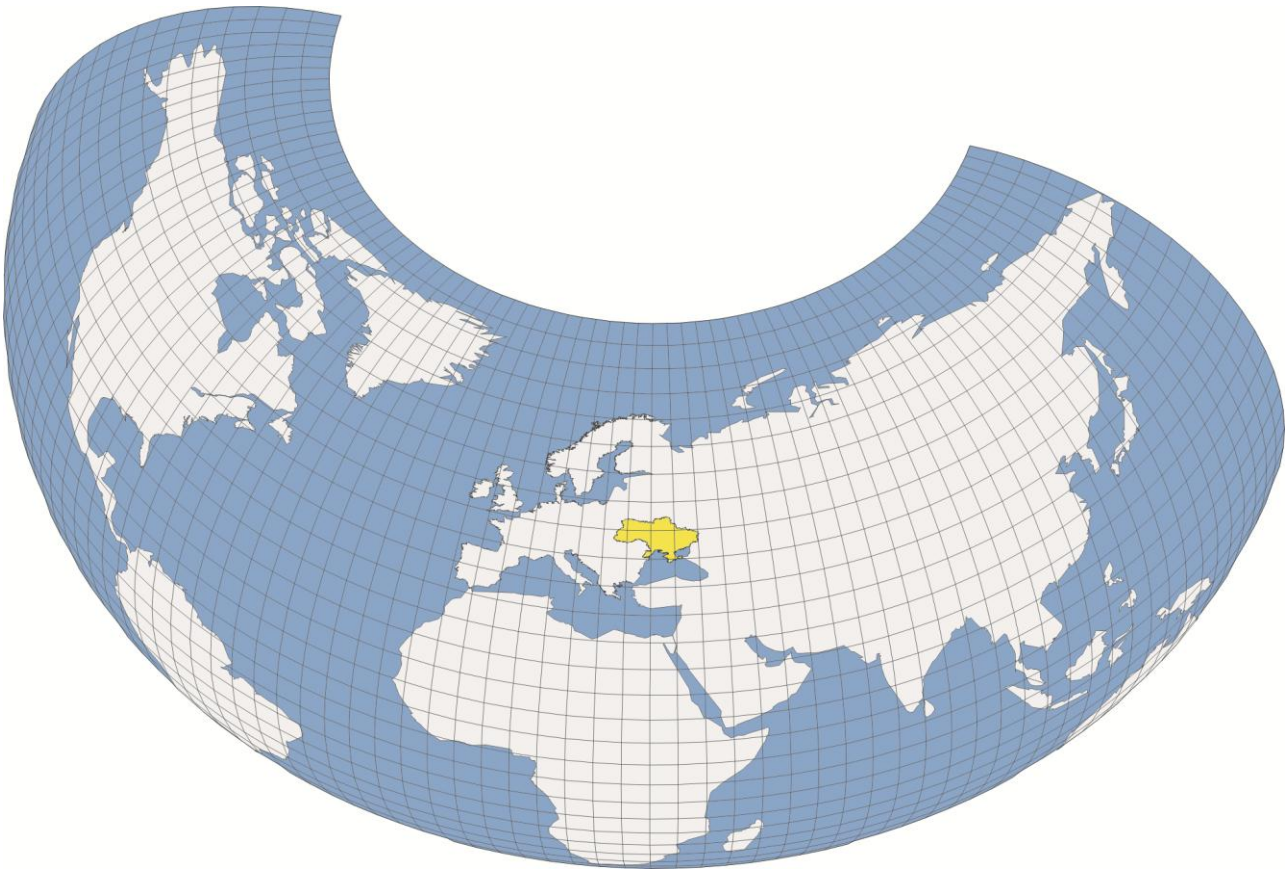

Fig.2. Modified isometric orthographic projection on the half of the torus
(adapted version: inclination of the torus $\beta = 48^{\circ}$, central meridian $\lambda_0 = 31^{\circ}\,\mathrm{E}$)

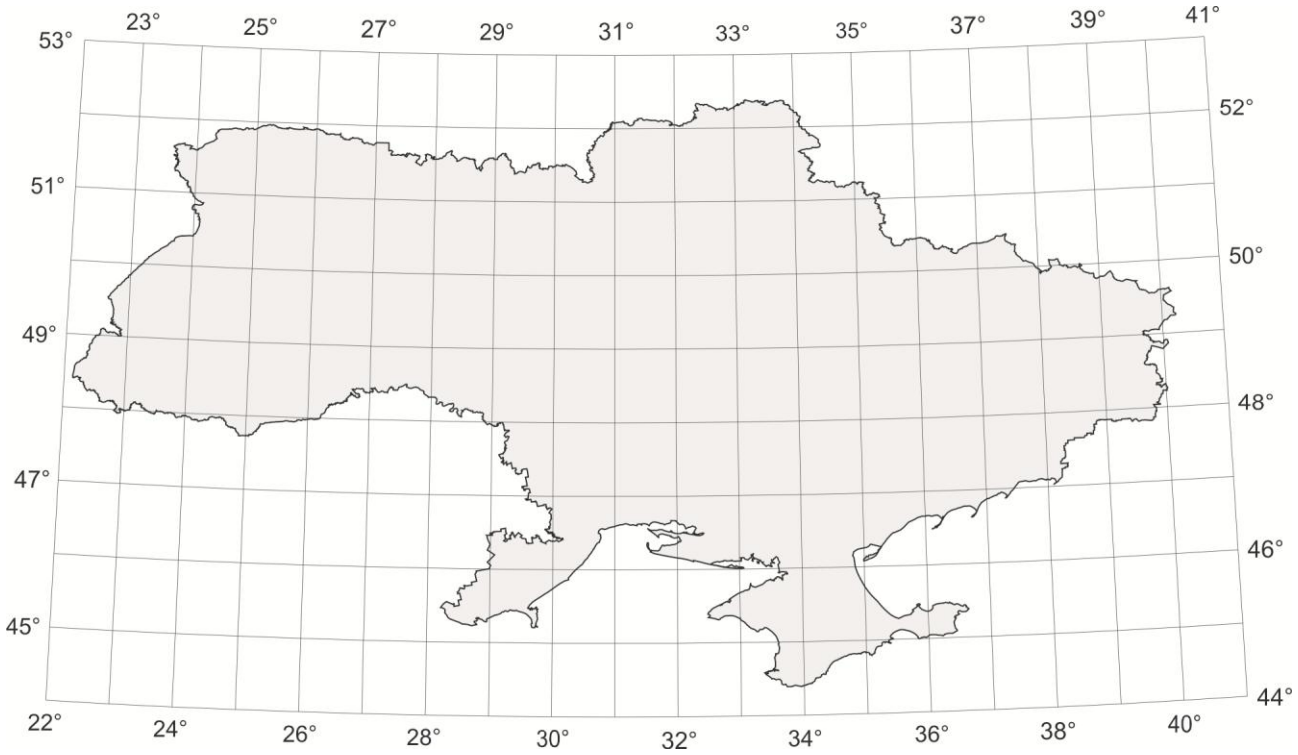

Fig.3. Map grid of modified isometric orthographic projection on the half of the torus
(adapted version: inclination of the torus $\beta = 48^{\circ}$, central meridian $\lambda_0 = 31^{\circ}\,\mathrm{E}$)

From other side, the stability of values of partial scales of lengths along the meridians *m* all over the field of map (tab. 2) provides the quasiequidistance of image, which allow to use the modified variant of projection for developing of mathematical bases of geographical maps of Ukraine.

The values of partial scales of lengths along parallels *n* (tab. 3) almost coincide with the values of partial scales of areas *p*, that is predetermined by minimal parameters of distortion of angles and lengths along the meridians.

Table 1

The values of maximal distortion of angles ε in the nodes of mapping grid of modified isometric orthographic projection

Latitude, φ°	Longitude, λ°									
	23°	25°	27°	29°	31°	33°	35°	37°	39°	41°
53°	34’25”	25’54”	17’18”	08’40”	00’00”	08’40”	17’18”	25’54”	34’25”	42’50”
52°	31’12”	23’29”	15’42”	07’51”	00’00”	07’51”	15’42”	23’29”	31’12”	38’49”
51°	28’00”	21’05”	14’06”	07’03”	00’00”	07’03”	14’06”	21’05”	28’00”	34’50”
50°	24’49”	18’42”	12’30”	06’16”	00’00”	06’16”	12’30”	18’42”	24’49”	30’52”
49°	21’40”	16’19”	10’55”	05’28”	00’00”	05’28’	10’55”	16’19”	21’40”	26’55”
48°	18’31”	13’58”	09’21”	04’41”	00’00”	04’41”	09’21”	13’58”	18’31”	22’59”
47°	15’22”	11’36”	07’46”	03’54”	00’00”	03’54”	07’46”	11’36”	15’22”	19’03”
46°	12’15”	09’15”	06’12”	03’07”	00’00”	03’07”	06’12”	09’15”	12’15”	15’09”
45°	09’07”	06’55”	04’39”	02’20”	00’00”	02’20”	04’39”	06’55”	09’07”	11’15”
44°	06’00”	04’35”	03’05”	01’33”	00’00”	01’33”	03’05”	04’35”	06’00”	07’11”

Table 2

The values of partial scales of lengths m in the nodes of mapping grid of modified isometric orthographic projection

Latitude, φ°	Longitude, λ°									
	23°	25°	27°	29°	31°	33°	35°	37°	39°	41°
53°	0.9819	0.9818	0.9817	0.9816	0.9816	0.9816	0.9817	0.9818	0.9819	0.9821
52°	0.9851	0.9849	0.9849	0.9848	0.9848	0.9848	0.9849	0.9849	0.9851	0.9852
51°	0.9879	0.9878	0.9877	0.9877	0.9877	0.9877	0.9877	0.9878	0.9879	0.9880
50°	0.9905	0.9904	0.9903	0.9903	0.9903	0.9903	0.9903	0.9904	0.9905	0.9906
49°	0.9927	0.9926	0.9926	0.9926	0.9925	0.9926	0.9926	0.9926	0.9927	0.9928
48°	0.9947	0.9946	0.9946	0.9945	0.9945	0.9945	0.9946	0.9946	0.9947	0.9947
47°	0.9963	0.9963	0.9962	0.9962	0.9962	0.9962	0.9962	0.9963	0.9963	0.9964
46°	0.9977	0.9976	0.9976	0.9976	0.9976	0.9976	0.9976	0.9976	0.9977	0.9977
45°	0.9987	0.9987	0.9986	0.9986	0.9986	0.9986	0.9986	0.9987	0.9987	0.9987
44°	0.9994	0.9994	0.9994	0.9994	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995

Table 3

The values of partial scales of lengths n in the nodes of mapping grid of modified isometric orthographic projection

Latitude, φ°	Longitude, λ°									
	23°	25°	27°	29°	31°	33°	35°	37°	39°	41°
53°	1.3290	1.3298	1.3304	1.3307	1.3308	1.3307	1.3304	1.3298	1.3290	1.3280
52°	1.3104	1.3111	1.3117	1.3120	1.3121	1.3120	1.3117	1.3111	1.3104	1.3094
51°	1.2928	1.2935	1.2941	1.2944	1.2945	1.2944	1.2941	1.2935	1.2928	1.2918
50°	1.2761	1.2769	1.2774	1.2778	1.2779	1.2778	1.2774	1.2769	1.2761	1.2752
49°	1.2604	1.2612	1.2617	1.2620	1.2621	1.2620	1.2617	1.2612	1.2604	1.2595
48°	1.2456	1.2463	1.2468	1.2471	1.2472	1.2471	1.2468	1.2463	1.2456	1.2446
47°	1.2315	1.2322	1.2327	1.2330	1.2331	1.2330	1.2327	1.2322	1.2315	1.2306
46°	1.2181	1.2189	1.2194	1.2197	1.2198	1.2197	1.2194	1.2189	1.2181	1.2172
45°	1.2055	1.2062	1.2067	1.2070	1.2071	1.2070	1.2067	1.2062	1.2055	1.2046
44°	1.1935	1.1942	1.1947	1.1950	1.1951	1.1950	1.1947	1.1942	1.1935	1.1926

Conclusions

As a result of the researches mathematically proved the feasibility of use a modified orthometric arbitrary orthoapsidal projections Armadillo onto inclined and deploy torus as a mathematical base for small-scale mapping of Ukraine.

The perspective direction for further researches in this branch is to study the possibilities of using minimal surfaces of revolution or surfaces with zero average curvature, in particular catenoid and helicoid surfaces, as fragmentary isometric deformation of catenoid, for developing of mathematical bases of geographical maps.

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The possibility and feasibility of using apsidal surfaces of revolution as an intermediate in the transition from spherical surface to the plane of the map is proved. The complete analytical study of a modified isometric orthographic projection at half of tilted and deployed torus is performed, and the models of grids of map projection for maps of the world and Ukraine are constructed.