UDC 598:631.459

FRACTAL THEORY OF SOIL EROSION

V.Melnyk, O.Shostak

Lesia Ukrainka Eastern European National University

Keywords: soil, soil space, Cantor set, fracture mechanics.

Research problem statement and its value

An application of the principle of fractal geometry for soil erosion process study is important under modern conditions of land resources intensive use of. The main task is a theoretical thinking of the principles of fractal geometry application to study the destruction of soil.

A review of recent scientific research

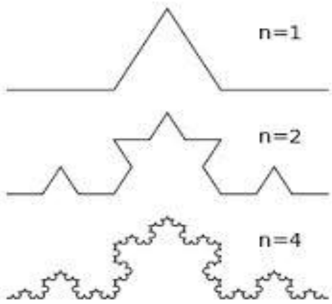
The works by Perfect E., Kay B., Tyler S. and Wheatcraft S. dedicated to the application of the fractal geometry principles in study of the soil erosion [5,6].

Thanks to the research of Michel R. and Garrison S. it has been proven, that the destruction of soil has the fractal nature [3,4]. Goldstein R.W., Masolov A.B. in their works have displayed, that to display a fractal model of soil erosion process is efficient to apply Cantor set [9].

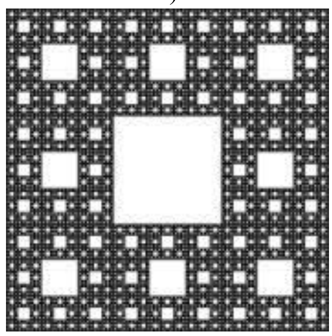
Presentation of the basic material

1. **The terms of fractal geometry.** The theoretical foundation of fractals was described in the well-known monograph by B. Mandelbrot «Fractal geometry of nature» [2]. At the base of the fractal geometry is the notion of dimension. Habitual for us topological dimension is always a whole number of points that lie along the smooth curve such as circumference as dimension 1 regardless of the space dimension where these points are represented. Opposite fractal dimension (Hausdorff –-Besicovitch) don't occur as equal. In nature it is connected not with topology but with metrics that means a constructible set [12]. Informal explanation is that for each number γ D_f – dimensional cube can be lay $N=\gamma^D$ into cubes like this by a factor of the similarity $r(N)=1/\gamma$. Fractal dimension *D* can be applied if an entire object can be divided into *N* parts each of which is similar to the output with similarity coefficient γ. Fractal structures are often viewed as chaotic disordered dynamic processes [7].

The concept of dimension has been long studied by mathematicians, but only when the monograph by Mandelbrot was published a concept dimension became widely used in physics, chemistry, geomorphology, geography [3,12]. Figure 1 shows examples of fractal Koch curve and Sierpinski square carpet.



a)



б)

Figure 1. Rules of construction a fractal Koch curve (a) and Sierpinski square carpet (b)

To Koch curve: m=4, P=3, that's why $D_f = \ln 4/\ln 3 \cong 1{,}2618$*;*

To Serpinski square carpet: m=8, P=3

$$D_f = \ln 8/\ln 3 \cong 1{,}8927$$

Sets for which Hausdorff dimension is more topological:

$$D_H > D_t \qquad (1)$$

In the work (8) the strong mathematical module dependencies of elasticity (E) from the fractal dimensions of void content (pores):

$$E/E_0 = (1-Q_V)^{V_{\ln k}} \qquad (2)$$

where $Q=\dfrac{\rho_1}{\rho_0}$ gravity of voids.

Value $V_{\ln k}$ is suggested to characterize by matrix 3x3:

$$V_{\ln k} = \begin{pmatrix} V_{100} & V_{011} & V_{111} \\ V_{010} & V_{101} & V_{111} \\ V_{001} & V_{110} & V_{111} \end{pmatrix} \qquad (3)$$

where V_{ijk} volume in the direction of the crystallographic axes (the Miller's indexes). For the different types of soil the numeric values of matrix $V_{\ln k}$ will be different and they fully characterize the elastic properties that mean their antierosion resistance.

2. Soil as fractal porous medium.

Fractal pore medium is such a medium in which there is a likeness of the pore medium and solid matrix (solid phase) [3]. According to the equation:

$$P_i \equiv V_i - V_{i+1} \qquad (i=0,....,m\text{-}1) \tag{4}$$

It means that the differential of volume of pore space P_i and partial volumes V_i are similar in form and correspond in scaling relation in size ratio. In general, soil samples can be represented as a set of voids and solids that is generally described as follows. For the introduction of large-scale properties we define the coefficient of the linear similarity r that is correlated with pore size p_i or partial volumes represented by their average (or median) diameters d_i :

$$p_{i+1} = rp_i \quad (r < 1) \tag{5}$$

$$d_{i+1} = rd_i \quad (r < 1) \tag{6}$$

Taken into consideration these obvious identities the differentials of pore space volume and partial volumes are scaled as r^3:

$$P_{i+1} = r^3 P_i \tag{7}$$

$$V_{i+1} = r^3 V_i \tag{8}$$

That implies:

$$P_i / V_i = \text{constant} = C \quad (i=0,....,m\text{-}1) \tag{9}$$

$$V_{i+1} = (1 - C_i) V_i \tag{10}$$

Porous coefficients during scalling of pore medium are homogeneous in all virtual fractions.

From the fractal properties of pore medium proceeds that for each virtual fraction porosity constant number N of smaller volumes V_{i+1} (or P_{i+1}) may be associated with V_i (or P_i) [2]. That means that each volume V_i contains N of smaller volumes V_{i+1} and one associated porous medium P_i :

$$V_i \equiv NV_{i+1} + P_i \quad (i=0,....,m\text{-}1) \tag{11}$$

In its turn, each volume V_{i+1} contains N of volumes V_{i+2} and one associate porous volume P_{i+1} and so on. So N of pore volumes can also be associated with every pore volume P_{i+1} and then the equation 11 takes such a form:

$$V_0 = \sum_{i=0}^{m-1} N^i P_i + N^m V_m \tag{12}$$

where $N^m V_m$ is the residual volume of the solid phase. Equation (10) takes the form:

$$V_{i+1} / V_i = (1 - C) / N \quad (i=0,....,m\text{-}1) \tag{13}$$

Accordingly:

$$V_m = V_{m-j} (1 - C)^j N^{-j} \quad (j=1,....,m) \tag{14}$$

In such a case the value of the porous medium is calculated from j to m. In a similar way the partial porous medium ϕ_i is calculated, that is defined by a partial volume V_i :

$$\phi_i \equiv (N^i V_i - N^m V_m) / N^i V_i \tag{15}$$

$$= 1 - (1 - C)^{m-i} \quad (i=0,...,m\text{-}1) \tag{16}$$

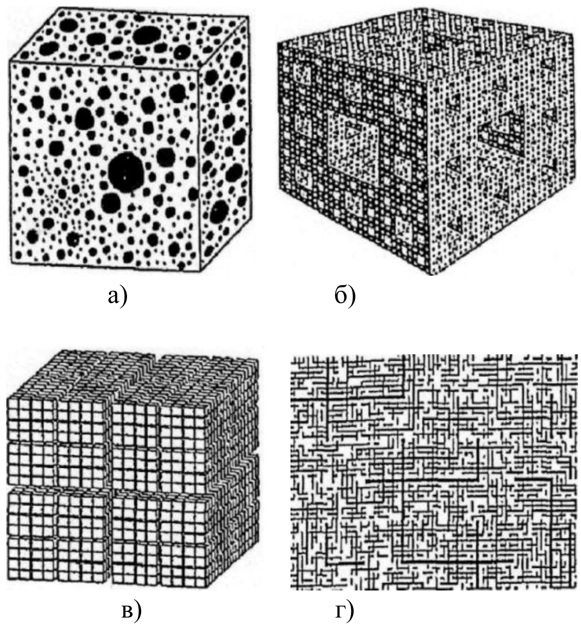


Figure 2. Conceptual models of porous media

a) porous media which is not necessarily the fractal; b) Menger sponge (ratio similarity r=1/3, N=2, fractal dimension D=2,7268); c) completely fragmented fractals of pore medium (r=0,476, number of the similar elements N=8, D=2,8; d) cross the intersection of random any fully fragmented fractal pore medium (r =0,485, the main part of the fractal correlation D=2,92, D=2,87), which shows the network and active faults.

The gained result proves that in fractal porous medium an index i reaches m respectively the partial volumes and pores size decrease that agree with the basic provisions of the fractaityl. In 3 – D measurable space of soil appears as self-similarity of cubes with dimensions (Figure 2) [3,4].

3. Virtual fractal model of soil erosion. Cantor set and his two and three dimensional modification (Sierpinski fractals) are often applied as the most ideal fractal structures [10]. Its entity is represented by Figure 3.

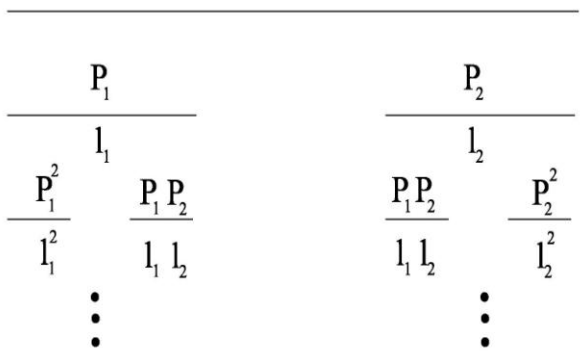


Figure.3. Cantor model

So here a unit segment has to be divided into two equal segments (l_1, l_2). Each segment, in turn, is divided into two and so to infinity. l_1 and l_2 correspond to different segments, and $l_1 l_2$ to surface mixed parts of the soil sample and thus provide the basis for calculating the fractal correlation. Each l has a certain amount of which is expressed by probability measure, in particular P_1 probability that breakpoint appears in the l_1, $P_1 P_2$ and in $l_1 l_2$ and etc.

According to such approach the algorithm of fractal dimension calculation is iterative [7] and involves solution of the equations system such as:

$$\begin{aligned}&\ln(n/m)\ln(l\ /\ l) - \ln(n/m-1)\ln(l) = \\ &q\left[\ln(P)\ln(l) - \ln(P)\ln(l)\right]\end{aligned} \tag{17}$$

$$\alpha = \frac{\ln(P_1) + (n/m-1)\ln(P_2)}{\ln(l_1) + (n/m-1)\ln(l_2)} \tag{18}$$

$$f = \frac{(n/m-1)\ln(n/m-1) - (n/m)\ln(n/m)}{\ln(l_1) + (n/m-1)\ln(l_2)} \tag{19}$$

$$(q-1)D_q = q\alpha(q) - f(q) \tag{20}$$

where n and m – number of each type of line segment; q – index; D_q – Reni dimension, that combines f and α . For the measurements l_1, l_2, P_1, P_2 and the set values q fractal graph in system $f-\alpha$ are calculated. An analysis of the equations (18-20) shows that if $q=0$, $f = D_0$, $q=\infty$, $\alpha = \ln P_1 / \ln l_1$ and when if $q=-\infty$, $\alpha = \ln P_2 / \ln l_2$. On the graph $f-\alpha$ it corresponds to the intersection D_∞ and $D_{-\infty}$ of abscissa a and characterizes the process of destruction mechanisms. Measurement of the linear parameters (l_1, l_2) and calculation of capabilities (P_1, P_2) can be done stereologically on a specially designed program for «Kwantimet 720» or simplified program «STIMAN».

Suppose that the process of destruction (breakdown) of the soil fragments, except the general topological ($D_t = 2$) is characterized by advanced fractal dimension ($2 \le D_f \le 3$), that directly takes the form of roughness and its distribution by size.

It is known that defective structures of soil fragment (cluster of tiny fractures and defects of the primary cluster, network of overloaded structural elements of the medium, fractures area and etc) have a certain similarity of structure and obey fractal laws of distribution and growth that is defective variety (of structure damage) develops in the body as similar fractal correlation cluster D_f ($0 \le D_f \le 3$).

It can be assumed, that the destroyed structure represents a fractal cluster of dimension D_f which coincides with the defective structure. In the destruction of one element of the structure characteristic of scale δ you need to consume energy ε_p and the total energy U_p, consumed in the destruction of the entire fractal structure can be evaluated in a such way [11]:

$$U_p = \varepsilon_p (L/\delta)^{D_f}, \tag{21}$$

where L – is typical body size.

An accumulated in the body energy before its destruction can be provided as:

$$U_e \approx \frac{\sigma}{2}\frac{2}{E}\sigma^3\left(\frac{L}{\delta}\right)^3, \tag{22}$$

where σ – significant voltage; E – modulus of elasticity. Next, if you count that all the accumulated energy is consumed on destruction, it's easy to get :

$$\sigma_p \approx \sigma_0 (L/\delta)^{(3-D_f)/2}, \tag{23}$$

where $\sigma_0 = (2E\varepsilon_p / \delta^3)^{1/2}$. The influence of body size on the resistance can be described the power law $\sigma_p \approx V^{-1/m}$, where V – the typical level of destruction.

If one compares the formula (22) and $\sigma_p \approx (L/3)^{-1/m}$ it will be got:

$$m = 6/(D_f - 3) \tag{24}$$

The value m is directly is related to a defective set of characteristics in particular with defective structure. When the value D_f is changed from zero (point defects, which are rarely found) up to three (entire volume is defective) m change from two to ∞, that fully agreed with experienced values m. Border $m \to \infty$ corresponds to the occasion $D_f \to 3$ if the defects are equally distributed throughout the three dimensional body volume [7].

It should be considered, that the calculation of the fractal surface ratio of destruction can be assessed in a large-scale effect that are characterized by geometric parameters of a defective structure in solid bodies, as well, as their properties and structure .

Conclusions and prospects of the further researches

1. Applying the principles of fractal geometry provides the analysis of antierosion soils resistivity.
2. The soil is considered as a medium which consists of a solid matrix and porous medium. The decrease of pores causes the reduction of their partial volumes that corresponds the provisions of the fractality.
3. To describe the virtual fractal model of the soil erosion the Cantor set may be applied. The more value of the fractal cluster the bigger defects (destruction) has a soil sample.
4. Further work will be focused on the application of the fractal dimension principles in evaluating of the level of degradation of soil parts and study soil erosion distribution process for the ratios of the processed samples.

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Fractal theory of soil erosion

V.Melnyk, O.Shostak

The issues of the of fractal geometry principles application for the identification and mapping of soil as a fractal porous medium are viewed. The possibility of use of Cantor set to describe the mechanics of soil fracture fragments is analyzed.