INVESTIGATION THE CHARACTERISTIC AND THE ACCURACY OF DTM CONSTRUCTION USING BICUBIC SPLINE INTERPOLATION

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Problem statement. One of main tasks of geodesy is supplying of humanity cartographical documents such as plans, terrain maps and topographic maps. With technological development the methods of relief visualization changed: from vividly graphical form of information cross over to modern electronic methods plus to the filled various GIS information. Major modification of modern maps is absence of scale of construction; the elements are vectorial and change in accordance with a select scale, that's why for the reflection of relief is not enough simple image of characteristic lines of relief and isohypses. There is a necessity of relief reflection by a surface [1,12]. DTM [2,13] is defined of space coordinates of points, which describe the difficult surface of topographical relief. On a method of the locations of points distinguish regular, irregular and structured models [1]. Present-day developments of mathematics permit the use of composite functions for description of general surfaces. The spline surfaces are the one of such general surfaces.

There is a problem of investigation the characteristic and the accuracy of DTM construction using bicubic spline interpolation (BSI). Authors are provided the usage of bicubic spline interpolation for the construction of DTM that more precisely represents relief and allows to construct surfaces on an automatic basis.

Analysis of the last researches and publications. There are several software systems that allow to construct the surfaces or find the value of heights in the necessary places of these surfaces such as Mathcad, Surfer, Mathlab [3,4,5]. But users are not known the principles of work of algorithms, which are used, and principles of finding bicubic polynomials for the calculation of these heights through closed of program code.

At the time of determination of many engineering tasks for elaboration of surfaces and calculation of accuracy of their construction it is necessary to work with a function that determined it. The problem of finding of this function appears therefore.

Analyze the scientific papers to this question we will separated [10], where made the smoothing over of data array, obtained by GPS method, using an incomplete bicubic spline, that takes into account 11 free parameters of polynomial, throw back last 5 numbers. As a result we got the program, which allowed to smooth out surface on a regular grid. The available accuracy under a mapping of relief was 0,35 m at the elevation changes of 20 m. Therefore our investigation were directed at the increase of accuracy of results: we proposed to use a complete bicubic spline, expecting 16 free polynomial parameters.

Problem definition. Development a software system for a receipt of complete bicubic polynomial for finding of heights of points of initial surface and establishment of accuracy of proceeding in relief by BSI.

Exposition of basic material. Depending on the types of data initial the bicubic spline can be presented in parametric representation in the form of Ermita, Bezier curves or B-splines [5,6,7,8].

It is known [11], that the surface patch $X = X(u,v)$; $Y = Y(u,v)$; $Z = Z(u,v)$; can be presented in parametric representation as:

$$Z(u,v) = a33u^3v^3 + a32u^3v^2 + a31u^3v^1 + +a30u^3 + a23u^2v^3 + a22u^2v^2 + a21 \quad (1)$$

that:

$$Z(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}u^iv^j \quad (2)$$

where *u*,*v* – parameters, which change in the certain fixed range, mainly $u\epsilon[0,1]$; $v\epsilon[0,1]$; $a33, \dots, a00$ - permanent coefficients within the limits of this surface, which can be incorporated in a matrix A_z a size 4x4. In future the function $Z = Z(u,v)$ will be examined, because the interpolation will be conducted for the heights.

Analogical expressions can be written down for $X(u,v); Y(u,v)$. Designate through A_x; A_y; A_z; the matrices coefficients at variables for expressions a size 4x4 $X(u,v); Y(u,v); Z(u,v)$; accordingly and enter the vectors of variables, $U = [1 \; u \; u^2 u^3]$, $V = [1 \; v \; v^2 \; v^3]$ then:

$$A_z = \begin{bmatrix} a00 & a01 & a02 & a03 \\ a10 & a11 & a12 & a13 \\ a20 & a21 & a22 & a23 \\ a30 & a31 & a32 & a33 \end{bmatrix}. \quad (3)$$

$$X(u,v) = U \times A_x \times V^T;$$
$$Y(u,v) = U \times A_y \times V^T;. \quad (4)$$
$$Z(u,v) = U \times A_z \times V^T.$$

The major task of construction of curved surface, which consists of bicubic areas there is finding of coefficients of bicubic polynomial A_x; A_y; A_z; inside every area for the coordinates of control points. Depending on type data out the equalization (1) cast in the form of Ermita, Bezier or B-splines. Let's consider these methods of presentation.

For the parameter representation of bicubic spline the initial data are 16 points. Set the known values of functions in 16 control points it is possible to make the next matrices of values:

$$X = \begin{bmatrix} X(-1,-1) & X(-1,0) & X(-1,1) & X(-1,2) \\ X(0,-1) & X(0,0) & X(0,1) & X(0,2) \\ X(1,-1) & X(1,0) & X(1,1) & X(1,2) \\ X(2,-1) & X(2,0) & X(2,1) & X(2,2) \end{bmatrix} \quad (5)$$

$$Y = \begin{bmatrix} Y(-1,-1) & Y(-1,0) & Y(-1,1) & Y(-1,2) \\ Y(0,-1) & Y(0,0) & Y(0,1) & Y(0,2) \\ Y(1,-1) & Y(1,0) & Y(1,1) & Y(1,2) \\ Y(2,-1) & Y(2,0) & Y(2,1) & Y(2,2) \end{bmatrix} \quad (6)$$

$$Z = \begin{bmatrix} Z(-1,-1) & Z(-1,1) & Z(-1,2) & Z(-1,2) \\ Z(0,-1) & Z(0,1) & Z(0,2) & Z(0,2) \\ Z(1,-1) & Z(1,1) & Z(1,2) & Z(1,2) \\ Z(2,-1) & Z(2,1) & Z(2,2) & Z(2,2) \end{bmatrix} \quad (7)$$

where $Z(0,0)$ - a value of height of point with coordinates $(u,v) = (0,0)$

$$u = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \quad (8)$$

$$v = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 2 \end{bmatrix} \quad (9)$$

From expression (3) we can get 16 linear equalizations with 16 unknown that do not make difficulties for finding of coefficients.

The bicubic polynomial can be presented as bicubic base functions (B-splines):

$$X(u,v) = U^{`} \times M_s \times P_x \times M_s^T \times V^{`T}$$
$$Y(u,v) = U^{`} \times M_s \times P_y \times M_s^T \times V^{`T} \quad (10)$$
$$Z(u,v) = U^{`} \times M_s \times P_z \times M_s^T \times V^{`T}$$

where $U^{`} = [u^3 \ u^2 \ u \ 1]$, $V^{`} = [v^3 \ v^2 \ v \ 1]$ - vectors of variables

$$M_s = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$ - base matrix, general for all B-splines

$$P_x = \begin{bmatrix} X(0,0) & X(0,1) & X(0,2) & X(0,3) \\ X(1,0) & X(1,1) & X(1,2) & X(1,3) \\ X(2,0) & X(2,1) & X(2,2) & X(2,3) \\ X(3,0) & X(3,1) & X(3,2) & X(3,3) \end{bmatrix}$$

$$P_y = \begin{bmatrix} Y(0,0) & Y(0,1) & Y(0,2) & Y(0,3) \\ Y(1,0) & Y(1,1) & Y(1,2) & Y(1,3) \\ Y(2,0) & Y(2,1) & Y(2,2) & Y(2,3) \\ Y(3,0) & Y(3,1) & Y(3,2) & Y(3,3) \end{bmatrix} \quad (11)$$

$$P_z = \begin{bmatrix} Z(0,0) & Z(0,1) & Z(0,2) & Z(0,3) \\ Z(1,0) & Z(1,1) & Z(1,2) & Z(1,3) \\ Z(2,0) & Z(2,1) & Z(2,2) & Z(2,3) \\ Z(3,0) & Z(3,1) & Z(3,2) & Z(3,3) \end{bmatrix}$$

Form of Bezier' bicubic spline surface, using the Bernstein polynomial, will appear as follows:

$$R(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} B_i^3(u) B_j^3(v) P_{ij} \quad (12)$$

$$R(u,v) = \left(B_0^3(u) B_1^3(u) B_2^3(u) B_3^3(u)\right) \times \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \times \quad (13) \times \left(B_0^3(v) B_1^3(v) B_2^3(v) B_3^3(v)\right) \quad (14)$$

where P_{ij} - datum vertex, the coordinates of which are set

$$b_{i,n}(t) = C_i^n t^i (1-t)^{n-i} \quad (15)$$

$$C_i^n = \frac{n!}{i!\,(n-i)!} \quad (16)$$

Analysing the reduced splines's representation and theirs physical content, conclude, that spline passes through every regular grid point, which is defined for its construction only at a parametric method. It is the author's opinion that spline is the unique, that will truely represent the real surface, passing through every regular grid point. The data out the construction of relief or surface is a set of data derived from instrumental measurements. Results of such measurings is a value X, Y, Z, which obtained with such accuracy, that the errors of their determination can be neglected. It is the reason of choice exactly of parametric method.

After the substitution of $Z(u,v) = U \times A_z \times V^T$ for the values of the proper variables, we will make simplification and will get equation (2). As you can see from (2) for finding of function *Z(u,v)*) we must to find the coefficients of matrix of A_z (coefficients a_{00}-a_{a33}) . For these we will find the product $U \times V$, as a result we get the matrix 16x16. This matrix will always be identical for any surface, because interpolation will be conducted in the conditional coordinate system.

As a result there was created the program for the calculation of heights of points into a regular grid with the use of complete bicubic spline. The following algorithm is developed:

1) Import in the program the set of values of these instrumental supervisions and forming from them of regular grid; verification of grid regularity.

2) Whether is there the coordinates, entered by user within the limits of surface, which information of regular grid is on.

3) Selection 16 control points for which will be formed a polynomial. It depends on the coordinates of point, user-definable, in which conduct the interpolation.

4) Review calculation of coordinates of 16 grid points in the local coordinate system *u*(-1;2), *v*(-1;2).

5) Compilation 16 polynomials for grid points and finding the value of coefficients *a00-aa33*.

6) Setting a bicubic polynomial and values of conditional coordinates *u,v* get the eventual value of coordinate of Z(*u,v*).

Testing of algorithm and program from determination a polynomial for the indicated point was offered on the example of recreation of surface of typical relief which is taken from [9], - standards for determination of category of relief difficulty. The example of relief area with regular grid is given on Fig 1.

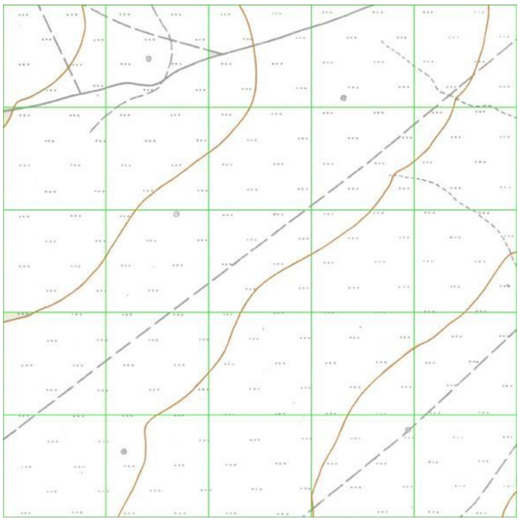
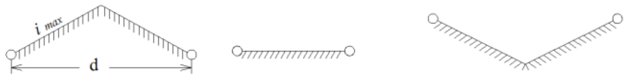


Fig 1. Relief with reference grid

The denseness of regular grid was chosen so that value of heights in grid point was possible to get by linear interpolation by hand. The practice been established that for achievement of satisfactory accuracy of approximation of relief in case of set of information about relief at the regular grid it is necessary to use linear interpolation exactly.

Let's explain this necessity.

Theoretically and practically, if the known heights in two nearly points of regular grid (built without the account of position of structural lines of relief: line of watershed or thalweg), that equally possible three variants of real position of earth's surface between them. And any analytical dependence with a 100% guarantee will not allow us to discover with what variant we have fact in this concrete case: *a*, *b* or *c* (Fig.2).



a) *b*) *c*)

○ -points of information collection

Fig. 2. Variants of relief recreation at the regular grid

To guarantee necessary accuracy of recreation of relief at such regular grid we can only in that case, when a denseness of grid will be very high. Approximately it is possible to calculated for obvious formula, which appear from Fig. 2.

(17)

where maximal error of determination of height of point in intervals between the grid point at the cost of approximation,

i_{max} - maximal earth grade, for hillside can accept 0,5.

For research there was taken the map scale of 1:5000 with contour interval of 5m, maximum earth grade 0,037 and the step of regular grid - 100m. After a formula (17) will get the maximally possible error of determination of height of point by linear interpolation that makes 3,7m, and MSE determination of height will make 1,23m.

The analysis of accuracy was suggested to conduct by the method of finding the variation of values of heights got by linear interpolation and obtained using the created program. On an initial surface by linear interpolation was found 125 points of heights. Using a polynomial the heights were calculated.

$$\begin{aligned} Z(u,v) = {} & 0{,}10613888 \times u^3 \times v^3 - 0{,}076833333 \times u^3 \times v^2 - \\ & - 0{,}33330555 \times u^3 \times v - 1{,}42109 \times 10^{-14} \times u^3 - \\ & - 0{,}122083333 \times u^2 \times v^3 - 0{,}0185 \times u^2 \times v^2 + \\ & + 0{,}488583333 \times u^2 \times v + 0{,}157 \times u^2 - 0{,}20355555 \times u \times v^3 + \\ & + 0{,}256833333 \times u \times v^2 + 0{,}574722222 \times u \times v + 2{,}199 \times u + \\ & + 0{,}10983333 \times v^3 + 0{,}012 \times v^2 - 2{,}916833333 \times v + 190{,}974 \end{aligned} \quad (18)$$

The values of coordinates and heights are obtained four methods and contained in the table 1. Taking the value, which got by linear interpolation as the ideal, were conducted calculation of MSE using the Gauss' formula:

$$m = \sqrt{\frac{\sum_{i=1}^{n} \Delta_i^2}{n}} \quad (19)$$

Table 1

Comparative table of the results of finding of heights and their errors

№	**X**	**Y**	**H** *Linear*	**Bicubic spline interpolation**		**Surfer 11**					
						Kriging		*Triangulation with Linear Interpolation*		*Modified Shepard's Method*	
				H	Δ	H	Δ	H	Δ	H	Δ
1	372,365	125,731	196,960	197,117	0,157	197,046	0,086	196,941	-0,019	197,140	0,180
2	375,089	181,446	195,862	195,912	0,050	195,850	-0,012	195,753	-0,109	195,986	0,124
…	…	…	…	…	…	…	…	…	…	…	…
124	323,121	18,066	198,341	198,116	-0,225	198,118	-0,223	198,041	-0,300	198,087	-0,254
125	317,160	79,336	196,711	196,728	0,017	196,733	0,022	196,621	-0,090	196,673	-0,038
				MSE =0,173		MSE =0,202		MSE =0,209		MSE =0,189	

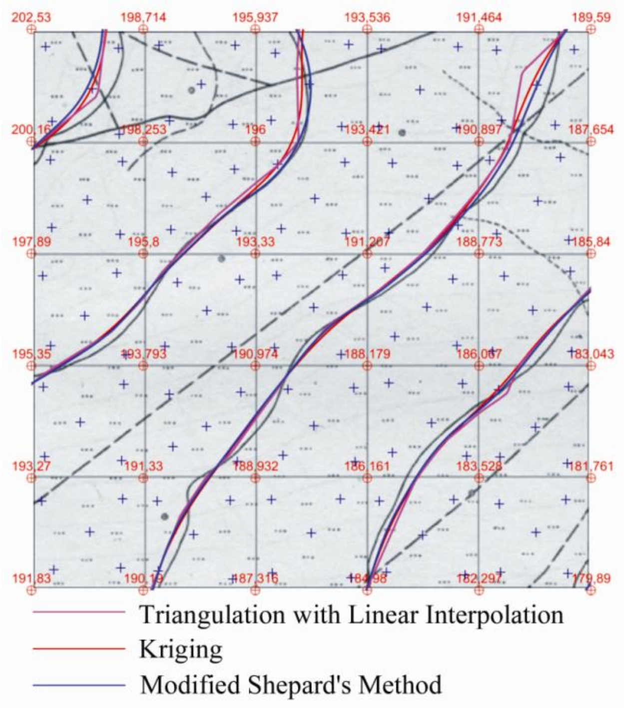


Fig.3 Composite map of the isohypses

Also authors was the conducted the researches of influence of elevation changes of surface on the value of error of finding of height. The values of elevation changes and rejections are given in a table 2

Table 2

The value of error of finding of height and elevation changes

№	Δ	H
1	0,156928	16,649
2	0,050109	15,551
…	…	…
124	-0,22506	18,03
125	0,0166	16,4
Mean	-0,06178	10,67738

From values (tabl.2) was got the Pearson correlation coefficient -0,126 with bilateral magnitude at the level of 0,161 for which the significant at a selection in 125 values are correlations higher as 0,1466. It denote the freedom from linear dependence between the elevation changes and error of finding of height. Spearman correlation, which is the non-parametric measure of dependence between two variables, was also calculated. Spearman's rank correlation coefficient between elevation changes and error of finding of height is 0,007. Such value the coefficient holds that is not dependence between the elevation changes and error of finding of height. It allows using the bicubic spline interpolation even at difficult relief. However, in such case authors recommend to use a regular grid with a greater denseness of points.

At the first term above represents method by authors was calculated the dependence remoteness the point outermost of surface, formed of 16 control points, which are data out for the calculation a bicubic polynomial. Value of linear dependence is -0,073 (the Pearson correlation coefficient), and value the Spearman correlation is -0,013. It show that points on surfaces placed independently in relation to the size of error of finding of height.

Conclusions

Conducting comparison of heights of the 125 points into a grid, obtained as a result of linear interpolation by hand and by bicubic spline interpolation, we got mean-square error reflection of relief - 0,1730 m. As see an error is very small in comparison with before calculated permissible value. That is why we can say that this interpolation method is precision and is usable for more precise recreation of the real surfaces in DTM.

The calculated values of correlation specify on freedom from influence of elevation changes of surface and remoteness the point outermost of surface and on the accuracy of finding of height by the bicubic spline interpolation method.

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Investigation the characteristic and the accuracy of dtm construction using bicubic spline interpolation

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The methods of presentation of bicubic spline are considered. Accuracy of reproducing of relief is investigational by BSI. Using statistical values proven that accuracy of interpolation by BSI does not depend on the elevation changes and from remoteness outermost of surface.