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MODELING THE SHAPE OF THE SURFACE AND ITS PROJECTION ON A PLANE: DELAUNAY TRIANGULATION AND ITS APPLICATIONS

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1. Basic concepts and properties

Consider a smooth surface S^2 immersed in Euclidean space $\mathbb{E}^3 = (\mathbb{R}^3; <.,.>)$ and α projection plane to this surface. On the surface $S^2 \subset \mathbb{E}^3$ points are inflicted "stakeout" and its projection onto the plane α; i.e. are inflicted points $(x_i, y_i) \in \alpha$, $i = 1, N$, and the value of the shape function $f : \alpha \to \mathbb{R}$ at these points, $f_i := f(x_i, y_i) \in \mathbb{R}$, 1, N.

The basic problem is - how to construct an approximation for the shape function $f : \alpha \to \mathbf{R}$, *which would in the best way to take into account the detailed shape of the surface $S^2 \subset \mathbb{E}^3$ and would be effective for practical calculations.*

Because the surface S^2 is in the general case quite irregular, the easiest method of Lagrange interpolation in this case is ineffective.
A better solution is a **Gaussian interpolation formula**:

$$f(x,y) = \frac{\sum_{i=1}^{N} f_i / [(x - x_i)^2 + (y - y_i)^2]}{\sum_{i=1}^{N} 1 / [(x - x_i)^2 + (y - y_i)^2]}, \quad (1.1)$$

wherein $(x, y) \in \alpha$. This means that the value of the function at any point $(x, y) \in \alpha$ is the mean value of the function at the points "stakeout", whereby - the weight with which each measurement point falls in the sum (1.1), is inversely proportional square of the distance for that point on the plane α. Equation (1.1) has the property boundary:

$$\lim_{x,y \to x_i,y_i} f(x,y) = f(x_i, y_i) \quad (1.2)$$

for all points $(x_i, y_i) \in \alpha$; $i = 1, N$. Gaussian interpolation (1.1) is, of course, the class C^∞; ie smooth, and its value are so important inequalities:

$$\min_{i=1,N} f_i \le \sup_{x,y \in \alpha} f(x,y) \le \max_{i=1,N} f_i \quad (1.3)$$

and

$$\sup_{(x,y)\in\alpha} |\Delta f| \le \max_{i=1,N} \Delta f_i , \quad (1.4)$$

what is the result of the linearity of the mapping (1.1). Nevertheless, interpolation (1.1) has a bad feature that at any point "stake out" derivatives $\partial f(x_i, y_i)/\partial x = 0 = \partial f(x_i, y_i)/\partial y$, $i = 1, N$. This means that the points "stakeout" are local extreme of the function $f : \alpha \to \mathbb{R}$, and their impact on these points is important in an area with a width of **1/N**.

In this case, the function $f : \alpha \to \mathbb{R}$ surface shaped like a hedgehog. In addition, the calculation is quadratic ally complexity dependent on the volume of data. And also, without additional assumptions on the actual function $f : \alpha \to \mathbb{R}$ Gaussian expression (1.1) does not have a reasonable limit when $N \to \infty$. At the same time it should be noted that an important property of a good interpolation method is the so-called "locality", that is, that if we supplemented our measurements additional point "stakeout" $(x_{N+1}, y_{N+1}) \in \alpha$; is the result of interpolation will be of little distinct from the previous outside the new point. Besides, since the method is linear Gaussian, it virtually completely ignores a property such surface $S^2 \subset \mathbb{E}^3$ as the curvature.

To the right you take into account the curvature require other methods of approximation of the surface, which would take into account the curvature explicitly. As is well known from the times of Euler, who first studied the problem and invented the so-called "cubic splines" for the modeling of curvature of the bent beam or rod. Unfortunately, this method, now well established, is of little use, because the result is a serious problem of local compliance spline-approximation for different neighboring areas in the plane, which is a problem of compliance is no longer on the collection points, and the boundary lines of these areas.

To alleviate the difficulties of this method often use the method of "smoothing" spline-interpolation using such functional minimization:

$$W_N(f) := \sum_{i=1}^{N} (f(x_i, y_i) - f_i)^2 + \lambda \int_\alpha (f_{xx}^2 + f_{yy}^2 + 2f_{xy}^2)\, dxdy , \quad (1.5)$$

where $0 < \lambda << 1$ and the integral in formula (1.5) is interpreted as the energy "deformation" of the surface $S^2 \subset \mathbb{E}^3$.

As the calculation of practical, spline-method (1.5) behaves well only for a sufficiently homogeneous surface. With this complexity computing becomes the fourth degree, which means that it reduces the grid step in the plane α twice the amount of computation time increases in sixteen! Besides, the solution is not very stable. For instance, at 100 x 100 grid algorithm is no longer usable.

Another *geodesic practical method* is to **approach Snellius of 1615** for the first time in the Netherlands, the laying of channels applied "triangulation" - that is, constructing triangular meshes for plane α and removing the corresponding function interpolation.

There are different approaches to the triangulation plane of measurement points on the basis of "staking". The most effective method of triangulation plane α is the Delaunay method using so-called areas Woronogo, who was at the beginning of the last century professor of mathematics at the University of Warsaw.

For constructing Woronogo areas we need to introduce a definition.

Definition 1. *Let the point* $(x_i , y_i) \in \alpha$, $i = 1, N$; *then the arrear* $V_i \subset \alpha$ *will be called the area Woronogo when it is the intersection all half-planes, containing this point, and limited median normals to segments connecting the point to all other points "stakeout".*

It is obvious that each area Woronogo will be called training ground, i.e. convex polygon. Then you can define a new vertex $w(A,B,C) \in \alpha$, which is the diametric intersection normalized segments AB і $BC \subset \alpha$, (P.S. Points $A,B,C \in \alpha$ – these are the points "stakeout") he will also belonging diametrical normalized to the segment $AC \subset \alpha$. Since the point $w(A,B,C) \in V_A \cap V_B \cap V_C$, it can be easily noted that the areas form a grid on Woronogo α, at each vertex of which there are exactly three edges and three respective areas Woronogo.

Then, when you connect with each other points in Woronogo, which are adjacent to each other, we get the Delaunay triangulation precisely, and vertices Woronogo areas are centers of circles described on the resulting triangles.

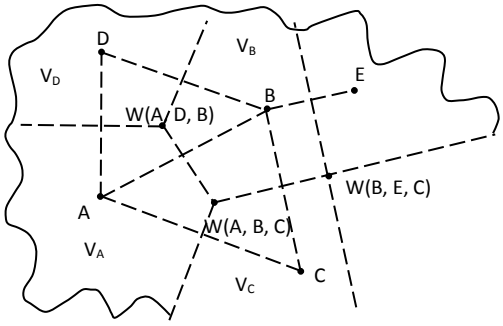
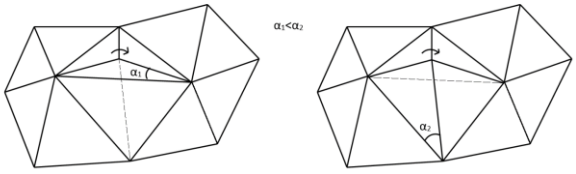


Fig.1. Woronogo areas and points of triangulation Delaunay.

Delaunay triangulation is very interesting and beneficial ownership: *Any triangulation plane Delaunay triangulation α will be when inside each circle described about the triangle is not any more vertices of the triangulation*.

This condition is necessary and sufficient for the Delaunay triangulation.

Fig.2. Added a new random point



Note 1. Interesting fact: *for points lying on the circle, any triangulation Delaunay triangulation will*!

Note 2. *The method of adding new points of triangulation is "robastic", ie fault-tolerant computing. Besides, the construction of the computational complexity is linear triangulation Delaunay to the number of points, which is an advantage too.*

From the point of view of computer calculations is also an important problem of identifying the Delaunay triangle in which there is a new triangulation point, and which, it turns out, is also solved algorithmically.

Also important is the problem of reconstructing the original Delaunay triangulation excludes certain triangulation of a set of points where they contain invalid information. It turns out that this problem is very soft and easily solvable - for this simply turn off unnecessary point of triangulation and Delaunay triangulations perform simple points only its neighboring.

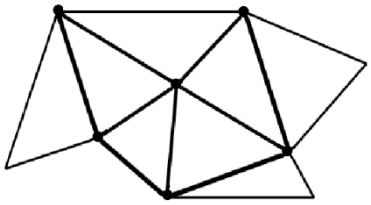


Fig.3. Elimination point of the triangulation Delaunay.

And another important advantage of the Delaunay triangulation - the possibility of a very simple combination of different triangulation, which have non-empty intersection, performing as usual Delaunay triangulation of a common area of intersection.

1.1. Delaunay Triangulation and interpolation problems.

Returning now to the problem of interpolation functions based on Delaunay triangulation, the next approach turns out to be computationally very efficient.

Definition 2. *Take the arbitrary point* $i = 1, N$ *Delaunay triangulation for a given plane* α *and define the points adjacent to this point as a reservoir of points* $N_i^{(1)}$ *the first level. Accordingly, the set of point of the second level of the* $N_i^{(2)}$ *– these are the points of the vertices of triangles that share a common edge of the triangles with a common vertex* $i = 1, N$.

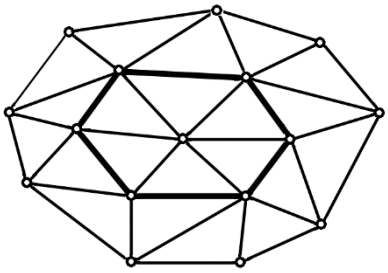


Fig.4. Collection points $N_i^{(1)}$ і $N_i^{(2)}$

The total amount collections of points $N_i^{(1)}$ and $N_i^{(2)}$ is equal to or greater than 6, i.e.

$$card\ N_i^{(1)} + card\ N_i^{(2)} \geq 6 \qquad (1.6)$$

for any vertex $i = 1, N$. In this case, it is possible to construct simple enough for each vertex $i = 1, N$ Delaunay triangulation function polynomial interpolation

$$f_i\ x, y\ :=\ \sum_{k+j=0}^{2} f_{kj}^{(i)} (x - x_i)^j (y - y_i)^k \qquad (1.7)$$

for certain parameters $f_{kj}^{(i)} \in \mathbb{R}$, $k+j = 0, 2$; in such a way that the values of the function (1.7) are equal to the values of measurements at all points from the set of $N_i^{(1)} \cup N_i^{(2)}$. You can also minimize the sum of squared deviations from the values of the function at the points "stakeout":

$$\min_{\{f_{kj}^{\,i} \in R:k+j=0,2\}} W_i\ f\ =\ _{j\in N_i^{(1)}\cup N_i^{(2)}}[f\ x_j, y_j\ -\ f_i]^2 u_j \quad (1.8)$$

the weights $u_j := 1 / |\Delta f_j|^2$; $j \in N_i^{(1)} \cup N_i^{(2)}$, are measurement errors of the function f $f : \alpha \to \mathbb{R}$ points "stakeout".

1.1. Metric tensor and tensor immersion shape of the surface in Euclidean space.

Consider a two-dimensional surface $S^2 \subset \mathbb{E}^3$, immersed in Euclidean space $\mathbb{E}^3$. Let the mapping of immersion will

$$r : S^2 \to \mathbb{E}^3, \quad (1.9)$$

and its local mutually unambiguous projection onto the plane $\alpha \subset \mathbb{E}^3$, tangent to the surface S^2, let us write the neighborhood of the point $A \in S^2$ in the form of projective

$$r = (x, y, z = f(x; y)^{\mathrm{T}}) \in \mathbb{E}^3, \quad (1.10)$$

where $(x, y)^{\mathrm{T}} := \alpha \in \mathbb{E}^2$; a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ is a function of the local shape of the surface S^2 satisfies the condition $f(0, 0) = 0$.

Now define two quadratic differential forms:

$$M(\alpha) :=< dr, dr >:=< d\alpha , g(\alpha)d\alpha >_{\mathbb{E}2} \quad (1.11)$$

form of so-called metric, and

$$Q(\alpha) :=< d^2r, n >:=< d\alpha , g(\alpha)d\alpha >_{\mathbb{E}2} \quad (1.12)$$

the so-called *Gaussian form the surfaced shape*, where $n \in \mathbb{E}^3$ is a vector *normal* to the surfaced S^2 at the point $A \in S^2$; is $n \perp S^2$ and $\|n\| :=< n, n >^{½} = 1$. Then, the surface curvature at the point S^2 in the direction of the single vector tangent to the surface S^2 at the point $A \in S^2$:

$$\xi := dr / dt\ 2\ T_A(S^2), \quad (1.13)$$

the limitations that meets the

$$< \xi , n >_{t=0} = 0, \quad \| \xi \| = 1, \quad (1.14)$$

is the curvature of the curve line $\gamma(\xi, n) \subset S^2$ intersection of the $\beta(\xi , n)$, passing through the point $A \in S^2$ and perpendicular to each other vectors $\xi, n \in \mathbb{E}^3$; with the surface S^2, that is $\gamma(\xi , n) := \beta(\xi , n) \cap S^2$; and will be equal to such a formula:

$$k\ \xi, n\ := \frac{M\ \alpha\ (\xi,\xi)}{Q\ \alpha\ (\xi,\xi)}\Big|_{t=0} = \frac{<\frac{d^2r}{dt^2}, n>}{<\frac{dr}{dt}\frac{dr}{dt}>}\Big|_{t=0} =< \frac{d\xi}{dt|_{t=0}}, n > \quad (1.15)$$

exactly equal to the *acceleration of the velocity vector* (1.13) at point $A \in S^2$ plane, passing through the velocity vector (1.13) and normal $n \in \mathbb{E}^3$ to the surface S^2 at point $A \in S^2$. Occur the next two fairly elementary theorems.

Theorem 1. *Footprint* $tr(qg^{-1})$ *and the determinant* $\det(qg^{-1})$ *are invariants of the curvature (1.15) at point* $A \in S^2$.

Theorem 2. *The matrix* $qg^{-1} \in End\ \mathbb{E}^2$ *has a diagonal character*

$$qg^{-1} = \begin{matrix} k_+ & 0 \\ 0 & k_- \end{matrix}, \quad (1.16)$$

where $k_- \le k_+$ *are the so-called main curvature of the surfaced at point* $A \in S^2$.

While the vector (1.13) can be chosen arbitrarily, it is with its help you can get to any point of a specially blended ambient point $A \in S^2$. Thus, by selecting a vector (1.13) as a vector of geodetic stream that reaches from the point $A \in S^2$ to the point $A(x, y) \in S^2$ shortest route can be used as the coordinates in the calculation of the optimal projection (1.10) in the speed variables $\xi \in \mathbb{E}^2$; the direction of the silent almost no restrictions, in addition to (1.14).

Suppose now that the plane $\beta(\xi, m) \subset \mathbb{E}^3$ also passes through the point $A(x, y) \in S^2$ and comprises a velocity vector $\xi \in \mathbb{E}^3$ and perpendicular to the unit vector $m \in \mathbb{E}^3$, that is $< \xi, m >_{t=0} = 0$, $\|m\| = 1$. Then it is quite easily see that the curvature of the curve $\gamma(\xi , m) \subset S^2$ intersection $\beta(\xi , m) \cap S^2$ will be equal to the formula

$$k(\xi, m) = k(\xi, n) < n, m >:= k(\xi, n)\ cos(n,\hat{}m). \quad (1.17)$$

The dependence of curvature (1.17) from the coordinates of the point $A(x, y) \in S^2$ gives the possibility of application of methods to construct the filter information of the noise measurements at various important characteristics, depending on the shape of the surface.

Thus, if the test object surface $S^2 \subset \mathbb{E}^3$ is described a function of the intensity $I : S^2 \to \mathbb{R}_+$; this filtering process can be described evolutionary diffusion equation

$$\partial I/\partial t + \mathrm{div}(D\nabla I) = 0, \quad (1.18)$$

wherein the diffusion coefficient is selected, or as $D := \exp(-\|\nabla I\|^2/\sigma^2)$, or as $D := \sigma^2/(\|\nabla I\|^2 + \sigma^2)$, $\sigma \in \mathbb{R}_+$ is a fixed parameter, and ∇I is the gradient, taking into account the shape of our face S^2. Then, equation (1.18) leads to a relationship of intensity $I = I(k)$ of curvature, which may already be used for other tests.

In other cases, the measurements of the intensity function $I : S^2 \to \mathbb{R}_+$ contain sharp changes in the amplitude, rather than the approach of diffusion (1.18) is used filtering method contained information from noise using the anisotropic Gaussian nuclei:

$$G_{\sigma_X,\sigma_I}\ X\ = G_{\sigma_X}\ X\ G_{\sigma_I}\ I\ X\ ,$$

$$G_{\sigma_\infty}\ X\ := \frac{1}{\overline{2\pi\sigma_X}} \exp\ -\frac{X^T X}{2\sigma_X}\ , \qquad (1.19)$$

$$G_{\sigma_I}\ I\ := \frac{1}{\overline{2\pi\sigma_I}} \exp\ -\frac{X^T X}{2\sigma_X}\ ,$$

where $\boldsymbol{X}$ **2** $\boldsymbol{End}(\mathbb{E}^2)$ is a matrix of pixel-grid coordinate measuring surface $\boldsymbol{S}^2 \subset \mathbb{E}^3$; and $\boldsymbol{\sigma_X}$ and $\boldsymbol{\sigma_I} \in \mathbb{R}_+$ are fixed parameters.

CONCLUSIONS

1. In short, the analysis methods for modeling the shape of a two-dimensional surface by its projection onto a plane gives enough grounds to say that the approach referenced on Delaunay triangulation is sufficiently effective, the face gradually stable and resistant to random errors measuring tool.

2. Being a geometric method is an important application opportunity generalizations of this approach to explicitly take into account the local curvature of the surface by an associated metric tensor and tensor of the surface shape of the surface immersed in Euclidean space.

3. Arguments are adduced effective use of geometric method for filtering measurement issues relevant functions of intensity, dependent on the shape of the object.

4. The method can be applied to a digital micro hard surface modeling bodies with the development of the images obtained from a scanning electron microscope.

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The analysis approximation-interpolation methods applied to modeling the shape of the surface and its projection on the plane. An analytical proposed approach to projection onto a plane based on Delaunay triangulation.